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Title of Thesis — Titre de la thèse

On the ~~Uniqueness~~ Problem of Uniqueness of Equilibrium Stationary General Relativistic Fluids

University — Université

University of Alberta

Degree for which thesis was presented — Grade pour lequel cette thèse fut présentée

Ph. D.

Year this degree conferred — Année d'obtention de ce grade 1980	Name of Supervisor — Nom du directeur de thèse Dr. W. Israel

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THE UNIVERSITY OF ALBERTA

ON THE PROBLEM OF UNIQUENESS OF EQUILIBRIUM STATIONARY GENERAL RELATIVISTIC FLUIDS

by

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A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

IN THEORETICAL PHYSICS
DEPARTMENT OF PHYSICS

EDMONTON, ALBERTA
FALL, 1980

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis enitled ON THE PROBLEM OF UNIQUENESS OF EQUILIBRIUM STATIONARY GENERAL RELATIVISTIC FLUIDS submitted by JAMES R. SAVAGE in partial fulfiment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

That a self-gravitating equilibrium perfect fluid in empty space has a spherical configuration if it is nonrotating or has a unique configuration characterized by some physical parameters if it is slowly rotating is considered physically evident but has not been rigorously derived from Einstein's field equations together with suitable asymptotic conditions. We therefore consider the equations for a general relativistic stationary spacetime which is asymptotically flat (and diffeomorphic to $\mathbb{R}^4$) and consists of an exterior vacuum solution and an interior perfect fluid in rigid motion with a given equation of state $\rho(p)$ satisfying $0 \le p \le \rho$, given total mass m and given surface temperature T_b. Global analysis techniques are used to show that there is no family of such static (nonrotating) spacetimes which depends differentiably on a parameter and contains the spherically symmetric solution except for families containing only solutions diffeomorphic to the spherically symmetric one. In the general stationary (rotating) case, if we require that the solutions be close to the static spherically symmetric ones (in the sense of a suitable topology on the set of stationary spacetime metrics) we can show that there is at most a one-dimensional differentiable family of global axially symmetric solutions parameterized by the value of the angular momentum J except for those obtained by coordinate

transformations. A recent result of Lindblom (1976) which shows that all such spacetimes are axisymmetric makes the latter result quite general.

ACKNOWLEDGEMENTS

I would like to acknowledge W. Israel for his encouragement and help.

To H.P. Künzle I owe two debts of gratitude. Firstly, this work is the result of the efforts of both of us and it was his experience that led us through the maze of possible approaches to the problem. Secondly, it is from his example and guidance that I have learned the methodology of researching a large problem.

I am grateful to the Natural Sciences and Engineering Research Council and the province of Alberta for partial financial support while this work was carried out.

NOMENCLATURE

Throughout this dissertation we will use geometrized units in which the speed of light $c=1$ and Newton's constant $G=1$. Greek indices will run from 0 to 3 except when they are used as n-tuple indices. The usage will make this clear. Small Latin indices will run from 1 to 3 while capital Latin indices will run from 1 to 2. Tensors will be written sometimes with and sometimes without indices when no danger of confusion exists. The set of reals will be denoted by $\mathfrak{R}$ and the integers and complex numbers by $\mathfrak{N}$ and $\mathfrak{C}$, respectively.

TABLE OF CONTENTS

CHAPTER PAGE

I. EQUILIBRIUM RELATIVISTIC FLUIDS 1

1.1 Introduction 1

1.2 Thermodynamic equilibrium 7

1.3 Stationary spacetimes 15

1.4 (2+1)-dimensional formalism 25

II. UNIQUENESS THEOREMS 31

2.1 Historical uniqueness theorems 31

2.2 Our approach 38

2.3 Spherically symmetric solutions 44

2.4 Equation of state 48

III. MATHEMATICAL PRELIMINARIES 56

3.1 Differential geometry in relativity 56

3.2 Banach spaces and infinite dimensional manifolds 61

3.3 Weighted Sobolev spaces 64

IV. ON THE UNIQUENESS PROBLEM FOR STATIC PERFECT FLUIDS 87

4.1 Introduction 87

4.2 Choosing a Banach manifold structure on the set $\{(\gamma, U, U_b)\}$ 89

4.3 Linearization of $\mathscr{L}$ to find $\ker \mathscr{L}'(\sigma)$ 98

4.4 Curves of solutions in $\mathscr{S}$ 110

V. ON THE UNIQUENESS PROBLEM FOR SLOWLY ROTATING STATIONARY FLUIDS 115
5.1 Introduction 115
5.2 A banach manifold of nearly spherical stationary spacetimes 122
5.3 Linearization of $\tilde{\mathscr{L}}$ to find ker $\tilde{\mathscr{L}}'(\sigma)$ 127
5.4 Curves of solutions in $\tilde{\mathscr{S}}$ 140
5.5 The problem of surjectivity 143

BIBLIOGRAPHY 147
APPENDIX 1. A LEMMA OF AVEZ 154
APPENDIX 2. SOLUTION OF THE LINEARIZED (2+1)-DIMENSIONAL STATIC FIELD EQUATIONS ON THE SPHERICAL BACKGROUND 158
APPENDIX 3. LINEARIZATION OF THE (2+1)-DIMENSIONAL STATIONARY FIELD EQUATIONS ON THE SPHERICAL BACKGROUND 163

CHAPTER I

EQUILIBRIUM RELATIVISTIC FLUIDS

1.1 Introduction

Consider a star. The word conjures up an image of a distant, hot radiating ball of matter. A closer look reveals many different types of stars with radically different compositions ranging from "ordinary" matter stars (which include the planets) to dense neutron stars. Among these there are many different populations with wide variances in luminosity, temperature, mass, angular momentum and density corresponding to differences in their internal structure such as differences in composition, nuclear and chemical reactions, convective motion and differential rotation. But all these quantities are determined by the star's history of evolution, starting perhaps with a gravitationally collapsing cloud of low density matter whose gravitational energy is converted into thermal energy, heating up regions in which different reactions which produce more heat can take place. The higher pressures thereby produced can slow or perhaps stop the collapse. But how can one determine a star's structure this way? Disney (1976) has suggested that the structure of an evolving protostar is very dependent on the initial conditions and numerical analysis of the gravitational collapse of clouds lends credence to this (c.f. Buff et al (1979)). Yet even though there is in some sense an infinite number of such initial conditions, when we look at the stars they seem to be very similar in their configurations, namely, they are almost spherical. For example, the sun, with an equatorial rotation period of about 25 days and the interior rotating slightly faster, has polar and equatorial radii that differ by only about 1 part in 10^5 (c.f. Hill and Stebbins (1975))

Physically, the reason is very intuitive. All nuclear and chemical reactions proceed towards their endpoints as the star evolves, with electromagnetic radiation carrying off any excess heat. Viscosity damps differential rotation. Perturbational analysis of a spherical star indicates that gravitational radiation carries away the energy of all but the $\ell=0$ and $\ell=1$ spherical harmonic components of the perturbation (c.f. Thorne (1967)). Thus a star gradually evolves towards some equilibrium, non-radiating body and can be described adequately by a perturbation of this equilibrium model for some latter portion of its life, even though its evolution is very complex.

But what constitution of materials and what spatial configurations can such an endproduct of stellar evolution have? It is well known that there are three main classes, the white dwarfs supported by electron degeneracy pressure, the neutron stars supported by neutron degeneracy pressure and black holes, the end result of total gravitational collapse.

The black hole equilibrium configurations are well understood. There is no internal structure to worry about and the "no hair" uniqueness theorems of Israel (1967) and Carter

(1971) show that their gravitational field is uniquely given by their mass m and, if rotating, their angular momentum J. (Similar "no hair" theorems hold if the black hole has a charge (c.f. Israel (1968)). However, even though most stars probably have magnetic fields and are neutral, dealing with charged stars or those with magnetic fields is not a trivial extension of the present work so throughout this dissertation we will consider only stars with no charge and no magnetic fields.)

We are here interested in the non-singular equilibrium stellar models. There are no such configurations for large masses (generally for masses greater than about two solar masses) but whether a star collapses totally depends on its equation of state and its history as well as its mass. Some large mass stars eject some mass before getting to the endpoint of evolution, thereby avoiding total collapse. We will therefore assume that the mass of a configuration is always in $[0, m_{crit})$ where m_{crit} corresponds to the critical mass where total gravitational collapse occurs. We can again ask the question, what are the equilibrium configurations of such stellar models? To determine all equilibrium solutions is quite hopeless. The purpose of this dissertation is to determine the general properties which any solution corresponding to a nonrotating or slowly rotating isolated equilibrium stellar model must possess.

Fortunately, as long as the density of matter is much less than 10^{49}gm/cm^3, which occurs for all the proposed realistic equations of state, the physical interactions in a star occur on two widely differing distance scales: the short scale nuclear and chemical interactions and the long scale gravitational interaction (c.f. Thorne (1967)). (This is not true for magnetic fields but as we noted before, we will not be dealing with this case.)

This enables the study of equilibrium stellar models to be divided into two parts: one studying the inter-particle interactions on a small scale to determine the constitution of cold (non-radiating) catalyzed matter, neglecting gravitation, and the second studying the large scale structure of the matter and gravitational field, treating the matter as a smoothed out fluid distribution. A review of small scale properties of matter for the first study is given in Zeldovitch and Novikov (1971). In the following we take up the second study for non-rotating or slowly rotating isolated stars. More specifically, we will investigate the global symmetries and the uniqueness of such configurations. Unfortunately the problem of existence of such configurations seems to be even more difficult. This will be mentioned occasionally with a fuller discussion of the difficulties being given at the end of the last chapter.

We start in the next section, then, with an investigation of thermodynamic equilibrium stellar models. It will be seen that global stationarity will be a reasonable assumption to make for such stellar models so we will then present some formalisms for stationary spacetimes which will be very useful for our later work. (Stationarity rules out any radiative processes.) The appropriate boundary conditions resulting from the assumption that we are dealing with an isolated system will also be given.

Although both non-rotating Newtonian stellar models (c.f. Lindblom (1978)) and non-rotating black holes (Israel (1967)) are spherically symmetric, i.e. they admit the group SO(3) as a group of isometries with the group orbits being spacelike 2-surfaces (which are then necessarily of constant positive curvature), this has not been proved for non-rotating equilibrium relativistic stellar models. It is intuitively so physically obvious though, especially because of the Newtonian and black hole results, that it has been assumed and used in many studies of stellar structure (c.f. Thorne and Zytkow (1977)) as well as in studies of gravitational collapse (for example, the Zeldovitch sequence, c.f. Harrison et al. (1965)). In the slowly rotating case the corresponding uniqueness theorems must be weaker in the sense that one expects uniqueness (with some physical parameters fixed) only for configurations "close" to spherical symmetry with small angular momentum. Thus they will be "local" not "global" results.

In chapter 2, then, we take a look at these historical uniqueness theorems and see what sort of directions one might try to take for the relativistic stellar model case. For reasons which will appear clearer later on, our results are "local" for both the rotating and non-rotating cases. They are thus weaker than the historical uniqueness theorems for the nonrotating case but are more or less equivalent to those for the slowly rotating case. In fact our results are valid for very general equations of state but cannot be used to increase the generality of the Newtonian slowly rotating case because the Newtonian case is more degenerate than the relativistic case. This will become clearer later.

We will then outline the approach which we will take towards proving that for a fixed equation of state, a fixed surface temperature and a fixed mass the corresponding isolated equilibrium solutions of the gravitational field equations are unique (up to coordinate transformations) provided the solution is "close" in a differentiable sense to a spherically symmetric solution. The reasons for the latter condition will become more apparent later. In particular, the solution will be spherically symmetric if it is not rotating and, if slowly rotating, will be uniquely determined by the small angular momentum. Since we must be "close" to spherical symmetry we then take a look at spherical spacetimes. This is followed by an investigation of the conditions on the equations of state which we must impose. It is seen that they are not very restrictive.

Chapter 3 consists of an introduction to the mathematical techniques which we use to obtain our results. Chapters 4 and 5 present the statements and proofs of the uniqueness theorems for the non-rotating and slowly rotating isolated equilibrium stellar models, respectively.

1.2 Thermodynamic equilibrium

In a fluid in which the reaction rates are too slow to be important on the time scales of interest the chemical composition of the fluid can be assumed to be fixed uniquely by two thermodynamic variables such as the number density of baryons n and the entropy per baryon s. This is obviously the case for a fluid in equilibrium so let us assume that we have such a simple relativistic fluid with viscosity and heat conduction. The fluid mechanics which we present here is standard and can be found, for example, in Misner, Thorne and Wheeler (1972) §22 and Weinberg (1972) p.53ff.

The stress energy tensor for such a fluid with unit four-velocity u^α ($u^\alpha u_\alpha=-1$) is

$$T^{\alpha\beta}= \rho\, u^\alpha u^\beta+(p-\zeta\theta)P^{\alpha\beta}-2\,\eta\,\sigma^{\alpha\beta}+2q^{(\alpha}u^{\beta)} \qquad (1.1)$$

where $\rho(n,s)$ is the total mass energy density in the rest frame of the fluid, $p(n,s)$ is the isotropic pressure in the rest frame, $\zeta(n,s)>0$, $\eta(n,s)>0$ are the bulk and shear viscosities respectively,

$$\theta:=\nabla_\alpha u^\alpha, \qquad (1.2)$$

$$\sigma_{\alpha\beta}:=\nabla_{(\alpha}u_{\beta)}+\dot{u}_{(\alpha}u_{\beta)}-\tfrac{1}{3}\theta P_{\alpha\beta} \qquad (1.3)$$

are the expansion and shear deformation rates respectively,

$(\; := u^{\nu}\nabla_{\nu})$. $P^{\alpha\beta} := g^{\alpha\beta} + u^{\alpha}u^{\beta}$ is the projection operator orthogonal to u^{α} and q^{α} is the heat flux. Round brackets around tensor indices will refer to symmetrization while square brackets will refer to antisymmetrization. The angular velocity tensor $\omega^{\alpha\beta}$ is given by $-P^{\alpha\mu}P^{\beta\nu}\nabla_{[\mu}u_{\nu]}$ We assume that q^{α} obeys the general relativistic version of the Fourier law of heat conduction, first proposed by Eckart,

$$q^{\alpha} = -\kappa P^{\alpha\beta}(\partial_{\beta}T + T\dot{u}_{\beta}) \qquad (1.4)$$

where $\kappa(n,s)>0$ is the coefficient of heat conduction.

Furthermore, we take over the standard thermodynamical and particle conservation laws. Baryon conservation implies

$$\nabla_{\alpha}(nu^{\alpha}) = 0. \qquad (1.5)$$

while the first law of thermodynamics is given by

$$d\rho = n^{-1}(\rho+p)dn + nTds \qquad (1.6)$$

where the temperature $T(n,s) := n^{-1}(\partial\rho/\partial s)_{n}$. As well as the third law, namely if $T=0$ then $s=0$, we require that all thermodynamic quantities are positive. Then the Bianchi identity together with the Einstein equations

$$\overset{4}{R}_{\alpha\beta} - \tfrac{1}{2}\overset{4}{R}g_{\alpha\beta} = T_{\alpha\beta} \qquad (1.7)$$

yield the conservation laws $\nabla_\alpha T^{\alpha\beta}=0$ which, when projected parallel and orthogonal to u^α, yield

$$\nabla_\alpha(\rho u^\alpha+q^\alpha)=-p\theta+2\,\eta\,\sigma^{\alpha\beta}\sigma_{\alpha\beta}+\zeta\theta^2-\dot{u}_\alpha q^\alpha$$

$$(\rho+p)\dot{u}^\alpha=-P^{\alpha\beta}\partial_\beta p+P^\alpha_\mu\nabla_\beta(\zeta\theta P^{\beta\mu}+\eta\,\sigma^{\beta\mu}-2q^{(\beta}u^{\mu)}).$$

(The convention we use for defining the Riemann curvature tensor is that used in Hawking and Ellis (1973) and is such that

$$\nabla_\alpha\nabla_\beta\chi^\mu-\nabla_\beta\nabla_\alpha\chi^\mu=\overset{4}{R}{}^\mu{}_{\nu\alpha\beta}\chi^\nu.)$$

These equations correspond to the Newtonian energy conservation law and Euler's equation respectively. The first can be rewritten as

$$T\nabla_\alpha(nsu^\alpha+T^{-1}q^\alpha)=\zeta\theta^2+2\eta\sigma^{\alpha\beta}\sigma_{\alpha\beta}+q^\alpha q_\alpha/\kappa T \qquad (1.8)$$

Defining the entropy four-vector to be $s^\alpha:=nsu^\alpha+T^{-1}q^\alpha$ the second law of thermodynamics takes the form $\nabla_\alpha s^\alpha\geq 0$ with equality iff the fluid is in thermal equilibrium. (In the fluid's rest frame $\nabla_\alpha s^\alpha$ is the rate at which entropy is being generated per unit volume.)

As Lindblom (1976) has noted, thermal equilibrium then implies the following properties. Since the fluid is shear free and expansion free, by (1.8), it is rigid in the sense that

$$P^{\alpha\mu}P^{\beta\nu}\nabla_{(\mu}u_{\nu)}=g^{\alpha\mu}g^{\beta\nu}\mathcal{L}_u(g_{\mu\nu}+u_\mu u_\nu)=0 \tag{1.9}$$

This notion of rigidity is equivalent to the definition given by Born, Herglotz and Noether: "A body is called rigid if the distance between every neighboring pair of particles, measured with respect to the world line of either of them, remains constant along the world line." (c.f. Trautman (1965)) If in addition there is no rotation, i.e. $\omega_{\alpha\beta}=0$, then $\nabla_\alpha u_\beta-u_\alpha\dot{u}_\beta=0$ so that u_α is hypersurface orthogonal,

$$u_{[\alpha}\nabla_\beta u_{\mu]}=0.$$

Furthermore, q^α vanishes (also by (1.8)) so the fluid is perfect, $\dot{u}_\alpha=-\partial_\alpha\log T$ and all thermodynamic variables are constant along u^α,

$$\dot{T}=\dot{n}=\dot{s}=0. \tag{1.10}$$

Euler's equation then becomes

$$d\log T=(\rho+p)^{-1}dp, \tag{1.11}$$

the equation of hydrodynamic equilibrium. Integrating this shows that there is an arbitrary integration parameter T_b, the value of the temperature at the surface of the star given by $p=0$, which does nothing but change the temperature of the star by a constant amount; so, when discussing the uniqueness of configurations we will always take T_b to be fixed. Using (1.11) in the exterior derivative of (1.6) it is easily seen that $dp\wedge d\rho=0$ so the fluid is barotropic, $\rho=\rho(p)$. Defining $\theta^\alpha=T^{-1}u^\alpha$ it is easily verified that the rigidity condition and (1.10) imply that θ^α is an infinitesimal symmetry of the spacetime structure and all thermodynamic variables in the region occupied by matter (but is not defined in vacuum).

An analogous result has been obtained by Stewart (1971) from a statistical mechanics approach. Using the relativistic Boltzman equation with the underlying physical assumption that the fluid is not too cold or dense, so that particles which are about to collide have uncorrelated momenta, he shows that in a collision dominated equilibrium, if one component of the fluid is not massless, the fluid must be rigid and the spacetime is locally stationary under θ^α.

It should be noted that if $T\equiv0$ so that the fluid is trivially in thermodynamic equilibrium, results analogous to those of Lindblom's above still hold. Namely, the fluid is rigid, barotropic with an equation of hydrodynamical equilibrium

$$d\rho=(\rho+p)d(\log n) \tag{1.12}$$

and $(\rho+p)^{-1}nu^\alpha$ is an infinitesimal symmetry of spacetime and all thermodynamic variables. Defining $T_e=n^{-1}(\rho+p)$ we then see that $dp=(\rho+p)d(\log T_e)$ and that $\theta_e^\alpha=T_e^{-1}u^\alpha$ is an infinitesimal symmetry of spacetime so all our results could be extended to cold matter at $T=0$ by substituting T_e and θ_e for T and θ respectively. Although cold matter is often assumed in astrophysical investigations, particularly of equations of state (c.f. Zeldovitch and Novikov (1971)), this is not an expected physical situation, especially in light of some formulations of the third law of thermodynamics (c.f. Pippard (1957) p.51). In the following we will assume $T>0$.

A stationary spacetime is defined as one which admits a globally timelike Killing vector field, along which all physical fields are Lie transported. Such a non-singular spacetime which admits a Cauchy surface and is asymptotically Euclidean (which we will define precisely later) can be shown to be in a state of thermodynamic equilibrium (Lindblom (1976)). Note however that this is not quite the converse of the preceding results since there we have only local stationarity in the region occupied by matter.

It would seem reasonable to conjecture that for an isolated equilibrium fluid in empty space the spacetime region outside the matter should be stationary as well as the interior region. This will clearly depend on the asymptotic conditions imposed, but is not easy to prove rigorously even if one demands pseudo-stationarity, i.e. existence of a global Killing vector field which is asymptotically timelike. However, that the conjecture is physically reasonable is enforced by a result of Friedman and Schutz (1975) which shows that stars with an ergosphere (a region in which the asymptotically timelike Killing vector field becomes spacelike) are unstable to a non-axisymmetric mode. On the other hand it is a stronger conjecture than one can justify in Newtonian theory (where stationarity implies all variables describing the fluid, including the velocity, are independent of time) since the non-axisymmetric rigid rotators such as the Jacobi ellipsoids are examples of non-stationary thermal equilibrium fluids. But in general relativity such objects would radiate gravitational waves and hence would not be expected to be in thermal equilibrium.

For solar sized black holes the time scales for approaching stationarity are of the order of milliseconds (Carter (1972)) so stationarity is a good assumption. Also, Thorne (1969) has shown that quadrupole vibrations in neutron stars are damped by gravitational radiation in the order of one second while Langer and Cameron (1969) have argued that many other vibrations are damped via nuclear reactions, which create thermal heating, on the same sort of time scale. However even after a star has reached its final state as a white dwarf or neutron star, the rate at which chemical and thermal equilibrium is reached and at which differential rotation is damped out is very slow. For example, in rapidly rotating white dwarf stars Kippenhahn and Möllenhoff (1974) have shown that differential rotation is damped out in a time scale of 10^5 years. This is slightly faster than the time scale of 10^6 years in which thermal equilibrium is approached. Having made the assumption of thermodynamic equilibrium however, we see that little is lost in assuming stationarity with ξ^α, say, being the global timelike Killing vector field.

The vector field Θ^α must either be proportional to ξ^α or Θ^α and ξ^α must be non-parallel commuting Killing vector fields. In the former case we have $u_{[\alpha}\xi_{\beta]}=0$ which in the case of a spacetime containing a perfect fluid (and obeying Einstein's equations) is equivalent to the material staticity condition

$$\xi_{[\alpha}\overset{4}{R}_{\beta]}{}^{\nu}\xi_{\nu}=0. \tag{1.13}$$

Lichnerowicz (1955) has shown that if the spacetime is asymptotically flat, asymptotically source free and topologically Minkowskian (which our stellar models are as we will see when we discuss boundary conditions and spherical solutions) then (1.13) is equivalent to the metric staticity condition

$$\xi_{[\alpha}\nabla_{\beta}\xi_{\nu]}=0 \tag{1.14}$$

which means ξ^α and hence Θ^α are everywhere orthogonal to a family of spacelike hypersurfaces by Frobenius' theorem.

In the latter case the interior of the star is axisymmetric. Lindblom (1976) has shown that by using the Cauchy-Kowaleski theorem and assuming that ξ^α is C^4, the spacetime is C^5 and the boundary of the star is smooth enough Θ^α can be extended analytically into the exterior region. This extension is linearly independent from and commutes with ξ^α and an asymptotic argument indicates that it must be a combination of a time translation and a rotation, so the spacetime is axisymmetric. Unfortunately it is difficult to rigorously match the above procedure with our boundary conditions so we will take the above result as an indication that there is little loss in generality when we assume a priori that our stationary, non-static, stellar models are axisymmetric, i.e. there exists a global one-parameter isometry group whose orbits are closed spacelike curves and which commutes with the timelike isometry group. A result of Carter (1970) shows that there is no loss in generality in assuming this commutativity. We will let η^a be the Killing vector field generator of the closed spacelike orbits.

From now on we will consider our stellar models to be isolated stationary thermal equilibrium solutions of Einstein's equations which are either static or axisymmetric. We have seen that they must then be perfect, rigid, barotropic fluids. In the next section we will present a convenient formalism for writing the equations governing such fluids and present the boundary conditions one imposes to ensure that the fluid is isolated.

1.3 Stationary spacetimes

Let (M,g) be a stationary spacetime. For simplicity let us assume that M is C^∞. The differentiability of g across the star boundary will depend on the form of the equation of state

$\rho(p)$ as $p \longrightarrow 0$ and its differentiability in the interior will depend of course on the differentiability of $\rho(p)$ with respect to p for $p>0$. We will see that we can assume the Lichnerowicz (1955) junction conditions, namely that g is at least C^1, piecewise C^3 (the second and third derivatives are continuous except at a finite number of hypersurfaces where they have finite limits on both sides). The metric will actually be C^3 in the exterior region so by a result of Müller zum Hagen (1970) it can be taken to be analytic in the exterior. Lindblom (1978) has shown that if $\rho(p)$ is analytic so is g in the interior (for rigid rotation). We also assume that ξ is C^0 piecewise C^2. With the Killing equations this implies that ξ is C^1, piecewise C^3.

The form for the metric g which we will derive below was originally given by Ehlers (1958) but the geometric derivation presented here first appeared in Künzle and Savage (1980a). Since M is stationary the flow of the vector field ξ corresponds to the orbits of a one-dimensional isometry group and these orbits are timelike C^2 submanifolds diffeomorphic to $\mathbb{R}$ since there are no closed timelike lines (causality condition). Let

$$e^{2U} := -g(\xi,\xi) \tag{1.15}$$

so that $\xi(U)=0$ and U is C^1, piecewise C^3.

We assume that the strong causality condition (Hawking and Ellis (1973), p.192), which has been shown to be valid for physically realistic solutions, holds for our stellar models. Namely, for all $x_0 \in M$ there exists a neighborhood V of x_0 that is intersected by each timelike orbit in at most one connected segment. This is true for the space of orbits Σ of ξ iff Σ has a manifold structure such that the canonical projection $\pi: M \longrightarrow \Sigma$ is C^2 (Palais (1957) p.20). Thus M is a principal fibre bundle over Σ with structure group $(\mathbb{R},+)$. (See for example Choquet-Bruhat et al. (1977b) p.128 for this and p.288 for the theory of connections used below.)

We orthogonally decompose T_xM into a 1-dimensional vertical subspace V_x parallel to ξ and a complementary horizontal subspace H_x by writing $\zeta \in T_xM$ as $\zeta = \zeta^0 \xi + \tilde{\zeta}$ with $\zeta^0 = e^{-2U} g(\zeta, \xi)$. This decomposition defines a connection since it is invariant under the group action of $(\mathbb{R},+)$. It can be equivalently characterized by a connection form, i.e., in this case a real valued 1-form ω that satisfies

$$\text{(i) } \mathcal{L}_\xi \omega = 0 \quad \text{(ii) } \omega(X_x) = 0 \Longleftrightarrow X_x \in H_x \quad \text{(iii) } \omega(\xi) = 1 \qquad (1.16)$$

These equations define ω uniquely if the horizontal subspace is given. The curvature form H associated to the connection form ω is then

$$H = d\omega + \omega \wedge \omega = d\omega$$

and $H = \pi^* \tilde{H}$ for a unique 2-form $\tilde{H}$ on Σ, since $\xi \lrcorner H = 0$ and

$\mathcal{L}_\xi H=0$ so that H is baselike. Similarly we have $U=\pi^*\tilde{U}$.

We define a Riemannian metric $\tilde{\gamma}$ on Σ by

$$\tilde{\gamma}(\tilde{\chi},\tilde{\zeta})|_{\tilde{x}}=e^{2U}g(\chi,\zeta)|_x \tag{1.17}$$

for any $x\in\pi^{-1}(\tilde{x})$, where $\chi,\zeta\in T_xM$ are the unique horizontal lifts of $\tilde{\chi},\tilde{\zeta}\in T_{\tilde{x}}\Sigma$, respectively. Then we find

$$g=-\pi^*(e^{2\tilde{U}})\omega\otimes\omega+\pi^*(e^{-2\tilde{U}}\tilde{\gamma}). \tag{1.18}$$

If (M,g) is also static, then $\tilde{\xi}\wedge d\tilde{\xi}=0$ where $\tilde{\xi}=g(\xi)$ by (1.14). Using (1.16iii) it is seen that $\tilde{\xi}=-\pi^*(e^{2\tilde{U}})\omega$ so $\omega\wedge\tilde{H}=0$ which implies $\tilde{H}=0$. The converse is also true.

We now drop the tilde and consider only the three-geometries (Σ,γ,ω,U) assumed globally defined, C^∞ in the exterior region, C^2 except at the star boundary with γ a positive definite Riemannian metric, ω a connection form on an $\mathbb{R}$-principal bundle over Σ and $U\in C^1(\Sigma,\mathbb{R})$. Note that there is no reason for the closed 2-form H to be exact.

On a local neighborhood of $x\in M$ there exist coordinates $(t,x^i)=x^\alpha$ such that $\xi=\partial_t$, i.e. $\mathcal{L}_\xi g=0 \Longleftrightarrow \partial_t g_{\mu\nu}=0$. Letting

$$g_{00}=-e^{2U} \quad g_{0k}=-e^{2U}a_k \quad g_{kl}=e^{-2U}\gamma_{kl}-e^{2U}a_ka_l \tag{1.19}$$

we have that

$$\omega = dt + a_k dx^k \quad \gamma = \gamma_{kl} dx^k \otimes dx^l \quad H_{ij} := 2\partial_{[i} a_{j]} \tag{1.20}$$

We now use such local adapted charts to derive the three-geometry formulas. First we observe that for any vector field χ such that $\mathcal{L}_\xi \chi = 0$, $\chi^0 = g(\xi, \chi)$ and $\chi^i \partial_i$ are a scalar and vector field, respectively, on Σ. By writing all the tensorial equations with the 0 index lowered and the Latin indices raised, the a_k do not appear except through H_{ij}. We use γ_{ij} to lower and raise all Latin indices (which will always be considered to run from 1 to 3).

Einstein's equations for a perfect fluid,

$$\overset{4}{R}_{\alpha\beta} - \tfrac{1}{2}\overset{4}{R} g_{\alpha\beta} = T_{\alpha\beta} = (\rho + p) T^2 \Theta_\alpha \Theta_\beta + p g_{\alpha\beta} \tag{1.21}$$

then become in the 3-dimensional formulation

$$\begin{aligned} \overset{3}{R}{}^{ij} = 2\gamma^{ir}\partial_r U \gamma^{js}\partial_s U + \tfrac{1}{2} e^{4U} h^i h^j + (\rho + p) e^{-4U} T^2 \theta^i \theta^j \\ -(2p e^{-2U} + (\rho + p) e^{-4U} T^2 \theta^2) \gamma^{ij}, \end{aligned} \tag{1.22}$$

$$\Delta U := \gamma^{ij} \nabla_i \partial_j U = M - \tfrac{1}{2} e^{4U} h^2 + (\rho + p) e^{-4U} T^2 \theta^2, \tag{1.23}$$

$$\varepsilon^{ijk} \nabla_j (e^{4U} h_k) = 2(\rho + p) e^{-2U} v T \theta^i, \tag{1.24}$$

where $h^i := \varepsilon^{ijk} H_{jk}$, $h^2 := \gamma_{ij} h^i h^j$, $\theta^i := \Theta^i$, $\theta^2 := \gamma_{ij} \theta^i \theta^j$, $M := \frac{1}{2}(\rho + 3p) e^{-2U}$ and $v := -u_0 > 0$ so that $v^2 = e^{2U} + T^2 \theta^2$. On the other hand, Θ^β is a symmetry of the spacetime metric and all thermodynamic quantities iff

$$\mathcal{L}_\theta U=0 \quad \mathcal{L}_\theta \gamma=0 \quad \mathcal{L}_\theta p=\mathcal{L}_\theta \rho=\mathcal{L}_\theta T=0. \qquad (1.25)$$

The rigidity conditions together with Euler's equation take the form

$$\theta \lrcorner H+d(\upsilon T^{-1}e^{-2U})=0 \qquad (1.26)$$

which implies that

$$\mathcal{L}_\theta H=d(\theta \lrcorner H)+\theta \lrcorner dH=0 \quad \text{whence } \mathcal{L}_\theta h=0. \qquad (1.27)$$

Since H is closed we also have

$$\nabla_r h^r=0. \qquad (1.28)$$

If M is static $H\equiv 0$ and equation (1.24) implies $\theta^i=0$. Einstein's equations then reduce to

$$\overset{3}{R}_{ij}=2\partial_i U\partial_j U-2\tilde{p}\gamma_{ij} \qquad (1.29)$$

$$\Delta U= M \qquad (1.30)$$

where $\tilde{p}:=pe^{-2U}$ while (1.26) becomes

$$T=T_c e^{U_c-U} \qquad (1.31)$$

in the interior, where the subscript c will always refer to the value at the center. We will see later that for all the cases we are interested in there will be a uniquely determined center. The equation of hydrostatic equilibrium (1.11) then becomes

$$dp+(\rho+p)dU=0. \qquad (1.32)$$

We can now state the boundary conditions which are usually imposed to restrict to solutions which are isolated, namely, we demand that spacetime is asymptotically flat at spacelike infinity in the following sense (c.f. Lichnerowicz 1955). In terms of the three geometry,

(i) there exists a compact $K\subset\Sigma$ and a diffeomorphism $\varphi:\Sigma\backslash K\longrightarrow \mathbb{R}^3\backslash B$ where B is a closed ball centered at the origin;

(ii) with respect to the standard coordinate system on $\mathbb{R}^3$

$$\gamma_{ij}=\delta_{ij}+O(|x|^{-1}) \quad \partial_k\gamma_{ij}=O(|x|^{-2})$$

$$U=O(|x|^{-1}) \quad \partial_k U=O(|x|^{-2})$$

$$h^i=O(|x|^{-2}) \qquad (1.33)$$

where $|x|^2=\sum_{i=1}^{3}(x^i)^2$.

These conditions are implied by more sophisticated definitions (c.f. Geroch (1972) for the special case of $\Sigma\cong\mathbb{R}^3$ which we will see later will hold for the spacetimes we are interested in. (We do not require any conditions at null infinity since stationarity has already ruled out any dynamics.) With such asymptotic conditions Lichnerowicz ((1955) p.126 et seq.) has shown that in the static case (1.30) together with $\rho\geq0$, $p\geq0$, implies that $U \leq 0$ on Σ and if there is a compact region $D \subset \Sigma$ such that $\rho = p = 0$ in $V = \Sigma \setminus D$ then U has no maximum or minimum in V and no critical point in some neighborhood of ∞ unless the space is flat. Furthermore $D_c := \{x \in \Sigma | U(x) \leq c\}$ is compact for all $c < 0$ and $S_c = \partial D_c = U^{-1}(c)$ is diffeomorphic to the Euclidean 2 sphere for c sufficiently close to 0, i.e. Σ is asymptotically spherical.

If Σ is homeomorphic to $\mathbb{R}^3$ (we will see later that Σ is actually diffeomorphic to $\mathbb{R}^3$ for all the situations we are interested in) the Poincaré lemma (c.f. Choquet-Bruhat et al. (1977b) p.213) implies that the closed form H is exact. Thus, even though the a_i are not necessarily the components of a 1-form on Σ we can find a 1-form α on Σ such that $H = d\alpha$. This α is determined uniquely up to a closed 1-form which by the Poincaré lemma must be exact, i.e. α is determined up to the differential of a function f on Σ. Imposing a gauge condition $\nabla^i \alpha_i = 0$ would give us a unique α since then $\Delta f = 0$ which implies $f = \text{const}$. However, we will not impose this as an extra condition (the reason for this will be apparent later). Rather, the gauge freedom will be treated in a manner analogous to the coordinate freedom.

We now want to define the mass and angular momentum of our isolated stellar model. There is a difficulty in general relativity in introducing such physical quantities which describe the structure of a gravitating system as a whole. This results from the fact that in flat space such quantities are associated with the action of the Poincaré group but in a curved space the corresponding "translation subgroup" is much larger than the constant vector fields one has in flat space. This is due to the fact that one must admit as asymptotic Killing vector fields all those vector fields which are linear in position with coefficients that form a skew tensor ("rotation") so the translation subgroup includes all vector fields which added to this do not disturb its asymptotic behaviour. The faster the spacetime metric approaches the Minkowski metric the closer the asymptotic symmetry group will be to the Poincaré group, but the smaller the amount of information available from the asymptotic behaviour of the gravitational field will be. The general situation is therefore very difficult but in our particular case (stationary, axisymmetric, asymptotically flat and Σ homeomorphic to $\mathbb{R}^3$) we can define quantities which in an asymptotic expansion of the metric are seen to correspond to the physical mass and angular momentum (c.f. Misner, Thorne and Wheeler (1972) §19).

Ehlers (1965) has shown that the total gravitational mass of an asymptotically flat stationary system in which Σ is homeomorphic to $\mathbb{R}^3$ can be defined by

$$m=(1/4\pi)\int_{\partial D}(\nabla^i U+\tfrac{1}{2}e^{4U}H^{ik}\alpha_k)dS_i \qquad (1.34)$$

where dS_i is the normal surface volume element of ∂D and where D is any domain of Σ containing all the matter. It is easily verified from Einstein's equations (1.22–1.24), the rigidity conditions (1.26) and the asymptotic conditions that the value of the integral is independent of the choice of D as well as α. In the static case this reduces to the usual integral $m=(1/4\pi)\int_D M dV$ obtained from integrating the relativistic Poisson equation (1.30), where dV is the volume element of Σ (c.f. Ehlers et al (1962) p.68). A calculation shows that this mass agrees with that found from the ADM four-momentum for an asymptotically flat stationary spacetime (c.f. Jang (1979)) and also agrees with the coefficient of $-1/|x|$ in an asymptotic expansion of U, i.e.

$$U=-m/|x|+O(|x|^{-2}) \tag{1.35}$$

at infinity.

A general definition for the angular momentum in the general case is somewhat trickier. (See for example Geroch (1972) or Ashtekar and Streubel (1979).) However, in the axisymmetric case a formula corresponding to (1.34) can be found. Recall that η is the axisymmetric vector field on Σ, so

$$\mathcal{L}_\eta \gamma=0,\ \mathcal{L}_\eta U=0,\ \mathcal{L}_\eta \alpha=0 \text{ whence } \mathcal{L}_\eta H=0 \tag{1.36}$$

and in the asymptotic region we can take

$$\eta=x^1\partial_2-x^2\partial_1+O(|x|^{-1}) \tag{1.37}$$

in terms of the cartesian coordinates. By fixing η in this manner we are eliminating some of the coordinate freedom so we will expect the slowly rotating isolated equilibrium stationary stellar models to form a 1-parameter family rather than a 3-parameter family. Now, $\overset{4}{\eta}{}^{\alpha}=(0,\eta^{i})$ is a 4-Killing vector field that commutes with $\xi=\partial_t$ and is everywhere spacelike. Defining the angular momentum J by

$$16\pi J:=\int_{\partial D}\sqrt{-|\overset{4}{g}|}\,\overset{4}{\nabla}{}^{0}\overset{4}{\eta}{}^{i}\delta^{123}_{ijk}dx^{j}\wedge dx^{k}=\int_{D}\sqrt{-|\overset{4}{g}|}\,\overset{4}{R}{}^{0}_{\nu}\overset{4}{\eta}{}^{\nu}dx^{1}\wedge dx^{2}\wedge dx^{3}\quad(1.38)$$

where $|\overset{4}{g}|=\det(g_{\mu\nu})$. it is clear that J is independent of D as long as D contains all the matter. This can also be written as

$$J=(1/16\pi)\int_{\partial D}\{\tfrac{1}{2}H^{ik}(\eta_k+\tfrac{1}{4}e^{4U}\alpha_k\alpha_l\eta^{l})+\alpha_k(2\eta^{k}\gamma^{il}-\eta^{i}\gamma^{kl})\partial_l U\}dS_i \qquad (1.39)$$

which can be verified to be independent of the choice of α.

1.4 (2+1)-dimensional formalism

The formalism we present here, valid in the case where U has only one (nondegenerate) critical point, was used by Künzle (1971) in an analysis of the linearized static Einstein equations because it allowed a result for 2-dimensional Riemannian manifolds of positive curvature to be used. (This result, with a corrected proof, is given in appendix 1.) We need this formalism so that Künzle's analysis can be used as well as to aid in the analysis of the linearized stationary Einstein equations. When we look at the spherically symmetric solutions we will see that U then has only one critical point, which is nondegenerate and defines a unique center x_c such that $U(x_c)=U_c$, $\partial_i U(x_c)=0$ and $\partial_i\partial_j U(x_c)\neq 0$. For technical reasons we will be forced to consider only spacetimes which are "differentiably" close to the spherical solutions so that this will hold true for the situations we are interested in. We are then able to rewrite our equations in terms of the two-dimensional geometry of the equipotential surfaces which will all be Riemannian 2-spheres.

Let S^2 be an abstract 2-sphere and $i_c: S^2 \longrightarrow \Sigma$ be the imbedding map of S^2 into Σ such that $i_c(S^2)=U^{-1}(c)=:S_c$ for any $c\in(U_c,0)$. Then any Σ 1-form is characterized by $\varpi:=i^*\omega$ and

$$\omega^0:=i^*(\nabla U \lrcorner \omega). \qquad (1.40)$$

This decompositon extends to any Σ-tensor field since Σ is Riemannian. An intrinsically defined normal derivative to the S_c surfaces is given by

$$\bar{D}\varpi=i^*(|\nabla U|^{-2}\nabla_{\nabla U}\omega) \qquad (1.41)$$

while the second fundamental form of S_c is

$$\Omega=i^*(\nabla(|\nabla U|^{-1}\nabla U)) \qquad (1.42)$$

(c.f. Kobayashi and Nomizu (1963)).

To find the local coordinate expression let $(\bar{x}^A, A=1,2)$ be a chart of S^2 and $i_c : \bar{x}^A \mapsto (U=c, x^A=\bar{x}^A(c))$ (using $(U, \bar{x}^A)$ as a chart of Σ). Then for a 1-form

$$\bar{\varpi}_A=\omega_1\partial U/\partial\bar{x}^A+\omega_B\partial x^B/\partial\bar{x}^A=\omega_A.$$

$$\omega^0=i^*(\nabla U \lrcorner \omega)=i^*(\omega_i\gamma^{ij}\partial_jU)=\omega^1. \tag{1.43}$$

In particular

$$\gamma_{AB}=\bar{\gamma}_{AB} \quad \gamma^{00}=\gamma^{ij}\partial_iU\partial_jU=:W^{-2} \quad \gamma^0_A=\gamma^{ir}\partial_rU\gamma_{1A}=\partial_AU=0 \tag{1.44}$$

and $\bar{\gamma}_{AB}$ is used to raise and lower all indices of S-tensors. If we also write $\gamma_{1A}=P_A$ and $P^A:=\bar{\gamma}^{AB}P_B$ we obtain

$$(\gamma_{ij})=\begin{pmatrix} W^{-2}+P^{x}P_{x} & P_B \\ P_A & \bar{\gamma}_{AB} \end{pmatrix} \tag{1.45}$$

Although P^A does not transform as a tensor on S (if $\bar{x}^A \longrightarrow \hat{\bar{x}}^A = \hat{\bar{x}}^A(U,\bar{x})$), by treating $P^A\partial_A$ as a vector field on S we find that for any S-tensor $\bar{K}$ (with possible U-dependence),

$$D\bar{K}=\partial_U\bar{K}-\mathcal{L}_P\bar{K} \tag{1.46}$$

is invariantly defined since $\bar{D}\bar{K}-D\bar{K}$ can be expressed in terms of S-tensors. For example, if $\bar{\varpi}$ is a 1-form, $\bar{D}\bar{\varpi}_A-D\bar{\varpi}_A= W^{-3}\omega^0\partial_AW-W^{-1}\Omega_{AB}\bar{\varpi}^B$. In particular,

$$D\bar{\gamma}_{AB}=2W^{-1}\Omega_{AB}. \qquad (1.47)$$

Note though that when U has only one critical point, ∇U is a vector field that vanishes nowhere except at the center and is orthogonal to the surfaces S_c. It can thus be used to construct a global polar type coordinate system (U,x^A) on Σ such that $P_A\equiv 0$ so $D=\partial_U$. To simplify matters we will always consider this to be done when dealing with this (2+1)-dimensional formalism, but it is not necessary. By writing the 1-indices up and the A-indices down one gets the same equations which we derive below and Künzle's analysis, done with $P_A\neq 0$, is merely simplified a bit.

After some work it is found that the Einstein equations take the form

$$\begin{aligned}DW=&-\Omega+MW^{-1}-\tfrac{1}{2}e^{4U}(W^{-3}h^{0^2}+W^{-1}\bar{h}^2)\\&+(\rho+p)W^{-1}e^{-4U}T^2\theta^2\end{aligned} \qquad (1.48)$$

$$\partial_{[A}\bar{h}_{B]}=0 \qquad (1.49)$$

$$D\bar{h}_A=\partial_A(h^0W^{-2})-4\bar{h}_A+2(\rho+p)W^{-1}e^{-6U}vT\bar{\varepsilon}_{AB}\bar{\vartheta}^B \qquad (1.50)$$

$$\begin{aligned}D\Omega_{AB}=&2W^{-1}\Omega_{AC}\Omega^C_B-W^{-1}\Omega\Omega_{AB}-2W^{-3}\partial_AW\partial_BW\\&+W^{-2}\bar{\nabla}_A\partial_BW+\tfrac{1}{2}W^{-1}\bar{R}\bar{\gamma}_{AB}-\tfrac{1}{2}W^{-1}e^{4U}\bar{h}_A\bar{h}_B\\&-(\rho+p)W^{-1}e^{-4U}T^2(\bar{\vartheta}_A\bar{\vartheta}_B-\bar{\vartheta}^2\bar{\gamma}_{AB})+2pW^{-1}e^{-2U}\bar{\gamma}_{AB}\end{aligned} \qquad (1.51)$$

$$\bar{\nabla}_B \Omega^B_A - \partial_A \Omega = \tfrac{1}{2} W^{-1} e^{4U} h^0 \bar{h}_A \tag{1.52}$$

$$\bar{R} - \Omega^2 + \Omega_{AB}\Omega^{AB} + 2W^2 - \tfrac{1}{2} e^{4U}(\bar{h}^2 - W^{-2} h^{0^2}) = -2\tilde{p} \tag{1.53}$$

where $\bar{h}^2 := \bar{\gamma}^{AB}\bar{h}_A\bar{h}_B$, $\bar{\theta}^2 := \bar{\gamma}^{AB}\bar{\theta}_A\bar{\theta}_B$, $\Omega := \bar{\gamma}^{AB}\Omega_{AB}$, $\bar{\varepsilon}_{AB} := W\varepsilon_{1AB} = \sqrt{\bar{\gamma}}\delta^{12}_{AB}$, $\bar{R}_{AB} = \frac{1}{2}\bar{R}\bar{\gamma}_{AB}$ and $\bar{\nabla}$ is the covariant derivative with respect to the connection defined by $\bar{\gamma}_{AB}$. Using $\theta^0 = \theta^i \partial_i U = 0$, equation (1.25) becomes

$$\mathcal{L}_{\bar{\theta}} W = 0 \tag{1.54}$$

$$D\bar{\theta}^A = 0, \quad \bar{\nabla}_{(A}\bar{\theta}_{B)} = 0 \tag{1.55}$$

while equations (1.28) and (1.26) yield, respectively,

$$Dh^0 - W^{-1}h^0 DW - W^{-1}\bar{h}^A \partial_A W + \bar{\nabla}_A \bar{h}^A + W^{-1}\Omega h^0 = 0 \tag{1.56}$$

and

$$D(vT^{-1}e^{-2U}) = W^{-1}\bar{\varepsilon}_{AB}\bar{\theta}^A\bar{h}^B, \quad \partial_A(vT^{-1}e^{-2U}) = W^{-1}h^0\bar{\varepsilon}_{AB}\bar{\theta}^B \tag{1.57}$$

and equation (1.27) translates into

$$\mathcal{L}_{\bar{\theta}} h^0 = 0, \quad \mathcal{L}_{\bar{\theta}} \bar{h}_A = 0 \tag{1.58}$$

Note that, in fact, the Lie derivative with respect to $\bar{\theta}^A$ of any S-tensor vanishes.

In the static case Einstein's equations are just

$$DW=-\Omega+MW^{-1} \tag{1.59}$$

$$D\Omega_{AB}=2W^{-1}\Omega_{AC}\Omega^{C}_{B}-W^{-1}\Omega\Omega_{AB}-2W^{-3}\partial_{A}W\partial_{B}W$$
$$+W^{-2}\nabla_{A}\partial_{B}W+\frac{1}{2}W^{-1}\bar{R}\bar{\gamma}_{AB} \tag{1.60}$$

$$\nabla_{B}\Omega^{B}_{A}-\partial_{A}\Omega=0 \tag{1.61}$$

$$\bar{R}-\Omega^{2}+\Omega_{AB}\Omega^{AB}+2W^{2}=-2\tilde{p}. \tag{1.62}$$

CHAPTER II

UNIQUENESS THEOREMS

2.1 Historical uniqueness theorems

Now that we have developed some formalisms to deal with our isolated equilibrium (stationary) stellar models we would like to get an idea of the types of approaches that may lead towards the uniqueness results which we mentioned in the first chapter, namely that in the static case the solutions must be spherically symmetric and in the slowly rotating case, for a given mass m, surface temperature T_b, and equation of state $\rho(p)$ the solutions "near spherical symmetry" form a 3-parameter family parameterized by the angular momentum. To orient ourselves, we will look at the approaches used in the historical uniqueness theorems, then at the general features which any uniqueness theorem must have and finally, outline the type of approach which we will take.

Proving that a static perfect fluid solution is spherically symmetric in nonrelativistic theory is considerably simpler than in the relativistic theory. (Static here means that the velocity of the fluid vanishes.) The metric of $\mathbb{R}^3$ is assumed to be Euclidean so that the physics is described by one function, the gravitational potential U_N, which satisfies a Poisson equation and is connected to the pressure via the equation of hydrostatic equilibrium $dp+\rho\, dU_N=0$. The proof that the solution must be spherical if $\rho>0$ in some compact domain of $\mathbb{R}^3$ and if $U_N \longrightarrow 0$ as $x \longrightarrow \infty$ was first given by Carleman (1919) for the case of uniform density models and by Lichtenstein (1919) for arbitrary density models. Such a result can be shown to follow from the theorem that a rotating Newtonian stellar model must have a plane of mirror symmetry which is orthogonal to the rotation axis of the star. This latter theorem was first proven by Lichtenstein (1918,1933) for the case of uniform density stellar models in rigid rotation. It was generalized by Wavre (1932) to the case of stationary axisymmetric barotropic ideal Newtonian fluids in differential rotation and further generalized by Lindblom (1977) to the case of barotropic ideal fluids which have stratified flows ($\vec{v}=(v_x,v_y,0)$, say). Wavre's proof involved the Green's function for the Laplace operator in the gravitational field equations so it cannot be generalized to relativistic stellar models where the field equations are nonlinear. Lindblom (1978) had hoped that his proof could lead to a generalization. However, his proof relies on constructing chords parallel to the axis of rotation (z-axis, say) with end points on the same level surface of the potential U_N. Then a plane is taken through z_m, the midpoint of those chords with the maximum z component and all functions are decomposed into even and odd parts with respect to reflections through this plane. The maximum principle is then used on the odd part of the gravitational potential U_N^- which is shown to satisfy $\Delta U_N^- \geq 0$ for all $z \geq z_m$. Such a coordinate dependent proof is clearly difficult to generalize and no one has yet succeeded in acheiving that end.

In the slowly rotating Newtonian case the question of uniqueness does not seem to be as settled. A stationary viscous heat conducting Newtonian fluid stellar model must have a plane mirror symmetry and be axisymmetric. However most discussions of equilibrium configurations of slowly rotating stars in the Newtonian theory start with an assumption about the particular form of the star boundary, such as being ellipsoidal or perhaps more complicated, as well as with assumptions about the equation of state such as the matter being homogeneous (c.f. Chandrasekhar (1969)). Neither ellipsoidal configurations nor constant density make much sense in relativity. Lichtenstein (1933) has shown that the Maclaurin ellipsoids are the only constant density configurations in a neighborhood of the spherically symmetric configuration.

There is considerable hope for obtaining uniqueness results in general relativity as the general relativistic case is less degenerate than the Newtonian one. In fact it is the linearization of the "magnetic" part of the gravitational field h which does not vanish on the spherical background, as we will see, so that there is more hope of proving that there exists a finite dimensional family of slowly rotating configurations with a given equation of state, given mass and temperature. We will run across difficulties in the existence part but we do obtain a uniqueness theorem for small J. (The vector field h is called "magnetic" because it is divergenceless and its curl is related to a mass current as is seen from equations (1.28) and (1.24) respectively.)

It is considered physically evident that a static general relativistic stellar model must be spherically symmetric. Indeed, that it is true in the Newtonian case suggests that a contradiction must involve strong fields accompanied by their significant nonlinearities. But Israel's (1967) proof that static black holes must be spherical shows that the nonlinearities of strong fields are unlikely to prevent the spherical symmetry of nonsingular stellar models. More precisely, Israel's theorem states that a static, asymptotically flat, regular spacetime corresponding to a vacuum solution of Einstein's equations must be a positive mass Schwarzschild solution if (i) past and future event horizons (c.f. Hawking and Ellis (1973), p.129) exist and intersect in a connected compact spacelike topological 2-sphere and (ii) U has no critical points exterior to the event horizons. Hawking (1972) showed that condition (i) was necessarily satisfied even in the stationary case. Müller zum Hagen (1973) and later Robinson (1977), by a much simpler and more elegant proof, showed that condition (ii) could be removed.

The methodology of these proofs is to derive expressions from the field equations with the form of a divergence equaling a definite signed quantity. By integrating the divergences are converted into surface integrals, the one at infinity vanishing because of the asymptotic boundary conditions and the one at the central singularity thus having a definite sign. The inequalities obtained then yield the desired uniqueness result. In particular, Israel constructs some expressions from a (2+1)-dimensional formulation of the vacuum Einstein equations with the divergence part being the derivative of some scalars with respect to U. The inequalities obtained then imply that $\partial_A\rho$ and $\Omega_{AB}-\frac{1}{2}\Omega\overline{\gamma}_{AB}$ vanish. Robinson used a 3-dimensional formulation of the vacuum Einstein equations to obtain an expression containing $R_{ijk}R^{ijk}$ where

$$R_{ijk}=2\nabla_{[k}R_{j]i}+\frac{1}{2}(\gamma_{i[k}\partial_{j]}R)$$

is the conformal tensor which vanishes iff γ_{ij} is conformally flat (c.f. Eisenhart (1926) p.89). Since one can show that

$$R_{ijk}R^{ijk}=e^{-4U}[8W^4(\Omega_{AB}-\frac{1}{2}\Omega\overline{\gamma}_{AB})(\Omega^{AB}-\frac{1}{2}\Omega\overline{\gamma}^{AB})+\overline{\gamma}^{AB}\partial_A W\partial_B W]$$

(c.f. Lindblom (1978) p.107), a conformally flat solution of Einstein's equations must have W, the magnitude of the gravitational field strength, being a function of U only. (The proof actually broke up into two cases, one yielding that R_{ijk} vanished and the other yielding directly that W was a function of U only.)

Avez (1964) had shown using Morse theory that when W

was a function only of U the spacetime must be spherically symmetric (even for the nonvacuum case). (He had thought that he need not assume that U had only one (nondegenerate) critical point but there is an algebraic error in the proof of his lemma 2, p.297. Künzle (1971) generalized his result by showing this assumption could be eliminated.) In fact when W is a function only of U he showed that U can have only one (nondegenerate) critical point so that Σ is then diffeomorphic to $\mathbb{R}^3$. Both of these statements rely on a result from Morse theory that two surfaces S_{c_1} and S_{c_2} are diffeomorphic if there is no critical value of U in $[c_1, c_2]$ (Milnor (1963) p.12). Thus Robinson was done.

The stationary black hole case was attacked by Carter (1971) in a similar manner using an expression containing a divergence and signed quantities but with an important difference. This expression was obtained from the linearized equations, not Einstein's equations, so the proof is not global as the above black hole theorems were. In fact Carter's result states that the possible families of solutions of Einstein's vacuum equations which are stationary and axisymmetric and depend differentiably (at least C^1) on some parameters form discrete families which can depend on at most 2 parameters. Hawking (1972) showed that the geometry exterior to the event horizon was necessarily axisymmetric for stationary black holes and Robinson (1975) then showed that only one such family exists, namely the Kerr family with $J<m^2$.

As we will see later we also approach the problem through an analysis of the linearized equations and our result in the stationary case will be of the same rigor as the stationary black hole theorems. In other words, both are "local" rather than "global" results.

There have been two main attempts to prove that static stellar models are spherically symmetric. Lindblom (1980) has attempted to follow the approach of obtaining an appropriate expression from the field equations which contained a divergence term and $R_{ijk}R^{ijk}$ as well as other signed quantities. This proved possible to do for the case of constant density but is highly dependent on this assumption as all the expressions must contain the density (and possibly derivatives of it) in them. A fairly thorough search for such expressions for more general equations of state has been carried out but with no success (Lindblom (1978)). The other approach is the investigation of the linearized equations in the (2+1)-dimensional formalism done by Künzle (1971). Here he considers the problem of proving that the static spherically symmetric stellar models are isolated from other possible stellar models by proving that there are no non-trivial static perturbations of a static spherically symmetric stellar model which leaves the central pressure p_c and central gravitaional potential U_c unchanged. These are not the appropriate physical constraints to put on the perturbation but it will turn out in our approach that the above result can be related to a perturbation with the physical constraints of constant mass , constant surface temperature and fixed equation of state $\rho(p)$.

Before passing to an outline of our approach we should mention one other paper. Marks (1977) claims to prove the general result that static stellar models must be spherical but his proof is fallacious. He assumes without justification that the curvature of the equipotential surfaces is constant. We are unaware of any uniqueness or existence results for the case of slowly rotating, stationary stellar models.

2.2 Our approach

Let us first examine some of the general features which a "static stars are spherical" theorem or a "stationary stars with fixed m, J, T_b and $\rho(p)$ are unique" theorem must possess. That the matter must be a fluid which cannot support stresses is evident from considering a static cubic crystal or a solid dumbbell slowly rotating about the axis joining the two bulbs in an asymptotically flat space. In fact the proof must use the fluid matter assumption in some global way for some small amount of solid matter arranged in a lattice or the shell of a dumbbell can destroy the uniqueness because of the long range effects of the gravitational field. Thus the proof must depend on the fluid properties of the matter throughout the region occupied by it.

The boundary condition at spacelike infinity, namely asymptotic flatness, is also crucial. If it was replaced by a non–spherically symmetric boundary condition the static theorem would clearly fail and since one can only expect uniqueness in the stationary case for slowly rotating configurations which are close to spherically symmetric this type of theorem would also fail. Thus the proof must also use the asymptotic boundary conditions in an essential way.

Furthermore the differentiability across the star boundary must play a role. If U, γ and h are "too differentiable" the boundary conditions will be overdetermined making the existence of a solution unlikely. If they are "not differentiable enough" one may lose the uniqueness. It is physically clear that this differentiability must depend on the equation of state and later in this chapter we will analyze this in some detail.

What kind of proof can fulfil these criteria? As we have seen, the black hole uniqueness theorems use the boundary conditions at infinity by integrating over all of Σ and using divergence terms to convert some of the integrals into surface integrals. However here one does not have to deal with star boundary conditions or properties of the fluid. As we noted before and is evident from the expressions of this sort obtained by Lindblom (1978) any proof along these lines is going to be heavily dependent on the equation of state and likely to be possible for very few.

Despite the complications due to the presence of a source, however, there is a heuristic line of reasoning which gives some hope. If the equation of state is fixed and the field equations are supplemented by the conditions for rigid motion one obtains a system of equations whose linearization may be expected to be elliptic, once the coordinate freedom is factored out. The set of solutions of these linearized equations together with appropriate boundary conditions should then be finite dimensional.

Our approach, then, is essentially a continuation of the deformation method used by Künzle (1971), namely an analysis of the linearized equations of a static (stationary) perfect fluid with a fixed equation of state $\rho(p)$, fixed surface temperature T_b and fixed total gravitational mass m (and fixed angular momentum J) on an arbitrary background solution over $\mathbb{R}^3$ and reasonably close to the unique spherically symmetric solution with the same $\rho(p)$, T_b and m. This is done in the spirit of Fischer and Marsden's (1975a, 1975b) work on linearization stability. Linearization stability roughly means that for every solution of the linearized equations on some background solution there is a corresponding solution of the nonlinear equations. When this is not true the system is said to be linearization instable on that background.

We give here a general indication of the argument in the static case since the stationary case is very similar and will wait until we have discussed the mathematical machinery used before we get precise. First we want to restrict to the study of spacetimes with the appropriate asymptotic behaviour and the differentiability conditions appropriate to the equation of state $\rho(p)$, which we can consider fixed. (The generality comes in being able to use the same types of spaces for different equations of state with a parameter determining the differentiability.) From the asymptotic expansion $U=-m/|x|+O(|x|^{-2})$ it is apparent that we can also consider m to be an asymptotic boundary condition so that we can restrict to spaces which will contain all the solutions of mass m. We will make the appropriate choice of a Banach space X of 3 metrics and potentials on Σ with the desired differentiability and asymptotic behaviour from among the weighted Sobolev spaces (Cantor (1975a), (1975b)). Then, using Einstein's equations to write a nonlinear differential operator mapping between Banach manifolds,

$$\mathcal{L}:\sigma=(\gamma,U)\longmapsto(R_{ij}-2\partial_i U\partial_j U+2\tilde{p},\Delta U-M)$$

from X into a space of symmetric covariant tensors and functions which will be isomorphic to some suitable weighted Sobolev spaces, the solution set of Einstein's equations with the desired asymptotic and differentiability conditions is just $\mathcal{L}^{-1}(0)$.

The coordinate freedom can then be handled by the fact that the set of diffeomorphisms on Σ that approach the identity at infinity operate in a natural way by pull-back on X and that the corresponding orbits are submanifolds of X. If we could show that the set $\mathcal{L}^{-1}(0)$ is contained in the orbit through the spherically symmetric solution we would have proved that these solutions are obtained from the spherically symmetric one by diffeomorphisms, or in other words, that the set of classes of physically equivalent solutions consists only of the spherically symmetric one.

We do not prove this general a result, however, for several reasons. First, the tangent map $\mathscr{L}'$ of $\mathscr{L}$ is not surjective so, as we will see, we are unable to prove that $\mathscr{L}^{-1}(0)$ is a submanifold. We are thus unable to tackle the problem of existence of solutions and in the stationary case must assume that rigid, rotating perfect solutions with nonzero angular momentum exist. Secondly, in order to extend our analysis of the linearized equations on the spherical background to backgrounds away from spherical symmetry we must use a slicing theorem for the orbits of the diffeomorphism group to obtain an elliptic operator from the tangent map $\mathscr{L}'$ to which we can apply a theorem of Nirenberg and Walker (1973). The slice $\mathscr{S}$ obtained represents the set of equivalence classes locally and Nirenberg and Walker's theorem deals with operators which are differentiably close to some operator so both theorems force us to consider only solutions which are close, in the Banach topology sense, to the spherically symmetric solution. Furthermore, since Nirenberg and Walker's theorem is a proof by contradiction there is little hope of obtaining an upper bound on how far away from spherical symmetry our result is still valid and thus little hope in finding an upper bound on J, in the stationary case, for which the uniqueness result we obtain holds. There is little doubt that such an upper bound exists, though, since we expect branch points as in the classical Maclaurin–Jacobi–ellipsoid series.

With all these qualifiers, what can we actually show? Since the slice $\mathcal{S}$ represents the physically distinct spacetimes (in a one-to-one correspondence) we still have a physically interesting result if we restrict consideration to it. In the static case we show that there are no non-constant C^1 curves of solutions in $\mathcal{S}$ going through the spherically symmetric solution in $\mathcal{L}^{-1}(0)$ which have the same mass and T_b. (We already consider the equation of state to be fixed, the mass has been fixed by the choice of Banach space and T_b is just a constant which we will always consider to be fixed.) In the stationary case we show that there are no non-constant C^1 curves of solutions in $\mathcal{T}$ (the corresponding slice) which have a constant angular momentum J (and where again, $\rho(p)$, m and T_b have been fixed).

The spherical solutions are clearly very important in our framework so in the next section we will look at them. We also need to investigate the equations of state which are physically appropriate and determine what differentiability conditions they impose before beginning our study of the weighted Sobolev spaces.

2.3 Spherically symmetric solutions

As we have already noted, when the strength of the gravitational field W is a function only of U then U has only one (nondegenerate) critical point at the center x_c, $U_c=U(x_c)$. Σ is diffeomorphic to $\mathbb{R}^3$ and the spacetime is spherically symmetric. If M is spherically symmetric then W is clearly a function only of U so U has only one critical point unless M is flat. The stability of Morse functions with a finite number of critical points (c.f. Golubitsky and Guillemin (1973), p.72,79) indicates that the property of isolated nondegenerate critical points is an open one. Since as we have seen above we are forced to restrict to solutions which are close to the spherically symmetric solution (in at least a C^1 sense of close in the function space in which U lies) we can always consider U to have only one (nondegenerate) critical point and can take Σ to be diffeomorphic to $\mathbb{R}^3$ so standard Euclidean coordinates can be used.

The following results about spherically symmetric spacetimes are standard (c.f. Künzle (1971)). The line element of Σ can be written as

$$ds^2=W^{-2}(U)dU^2+r^2(d\theta^2+\sin^2\theta d\varphi^2) \qquad (2.1)$$

where θ and φ are angular coordinates, $r^2(U):=2/\overline{R}$ and the static field equations (1.59–1.62) imply

$$rd(W^2)+4W^2dr=2MrdU \qquad (2.2)$$

$$Wdr=(1+r^2W^2+\tilde{p}r^2)^{1/2}dU \qquad (2.3)$$

These can be integrated to give

$$W=m(U)/r^2 \qquad (2.4)$$

where

$$m^2(U)=2\int_{U_e}^{U}Mr^4dU$$

and $m(U)$ agrees with the total gravitational mass m in the exterior region. In the vacuum region (2.3) implies

$$U=-\sinh^{-1}(m/r)=-\log\left(m/r+(1+m^2/r^2)^{1/2}\right). \qquad (2.6)$$

Note that r is not the Schwarzschild radial coordinate r'. In this coordinate

$$U=\tfrac{1}{2}\log\left(1-2m/r'\right) \qquad (2.7)$$

in vacuum. We could also write the line element as

$$ds^2=A^2(r)dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2) \qquad (2.8)$$

where $A=W^{-1}dU/dr$ so that in vacuum $A=(1+m^2/r^2)^{-1/2}$ or in terms of a Riemannian normal coordinate system (x^i) based at the centre x_e which are related to the polar coordinates (r,θ,φ)

in the usual way, i.e.

$$ds^2=(\delta_{ij}-B(r)x_ix_j)dx^idx^j \tag{2.9}$$

Here $B=r^{-2}(1-A^2)$ which in vacuum is

$$B=m^2r^{-4}(1+m^2/r^2)^{-1}. \tag{2.10}$$

Having chosen a function $\rho(p)$ (subject to conditions which we will state in the next section) the equation of hydrostatic equilibrium $dp+(\rho+p)dU=0$ determines $p(U)$ (and hence $\rho(U)$, $\tilde{p}(U)$, $M(U)$) uniquely up to a constant which can, for example, be chosen to be the value U_b of U on the star boundary, i.e.

$$U-U_b=-\int_0^p(\rho(\bar{p})+\bar{p})^{-1}d\bar{p}. \tag{2.11}$$

Since U, dU/dr and p must be continuous, as is seen from Einstein's equations (1.29, 1.30) with the assumption that $\rho(p)$ is piecewise C^0, it follows that (2.2) and (2.3) can be integrated for $r(U)$ and $W(U)$ in terms of their values $r_b=-m/(\sinh U_b)$ and $W_b=mr_b^{-2}$ so that the whole solution is determined by the two constants m and U_b. (A subscript b will always refer to the value of a scalar on the surface of the star, even in the stationary case.) The center is then defined by that value, U_c, of U for which r vanishes. But since it is a critical point of U

we must also have $W_c=0$ which is likely to impose an additional condition, determining U_b in terms of m.

At least, this is what happens for specific very simple models, like the interior Schwarzschild solution (where $\rho(p)=\rho_0$ =const.). We conjecture therefore that to given $\rho(p)$ and m there exists a unique spherically symmetric static stellar model provided $\rho(p)$ satisfies the conditions given in the next section and $0\leq m\leq m_{crit}$. (Clearly m must be bounded from above to prevent the formation of a black hole, but as we noted in §1.1 the value of m_{crit} depends on the equation of state.) A proof along the above lines would have to depend on a very close analysis of the nonlinear system (2.2, 2.3) and may not be so easy to prove for a general equation of state. The theorem which we prove in chapter 4 is a linearized version of this statement.

If this conjecture is true then we have for every $\rho(p)$ a curve $m\longmapsto\sigma\in X$ which tends to the flat solution $(\gamma_{ij}=\delta_{ij},U=0)$ for $m\longrightarrow 0$. This curve will be continuous when we topologize X. Unfortunately, it fails to be differentiable at $m=0$. This fact prevents us from applying to this situation a powerful technique of Cantor (1979) which requires some elliptic operators to be "differentiably close" to ones with constant coefficients. (This theorem is theorem 3.7 of the next chapter. We will discuss this more later when we have developed the tools to do so.)

2.4 Equation of state

We now want to consider the conditions on the equation of state $\rho:[0,p_c]\longrightarrow\mathbb{R}: p \longmapsto \rho(p)$ which are sufficient to prove our uniqueness theorems for stellar model configurations with fixed m, T_b, $\rho(p)$ and J in the static and stationary cases. In addition to the physically obvious conditions $0\leq p<\infty$, $0\leq \rho<\infty$ made in the thermodynamic argument, we also make the standard assumption that $p\leq\rho(p)$. This is implied by Hawking's dominant energy condition, namely, for every timelike vector field χ^α, $T^{\alpha\beta}\chi_\alpha\chi_\beta\geq 0$ and $T^{\alpha\beta}\chi_\beta$ is a non-spacelike vector. This can be interpreted as saying that to all observers the energy density appears non-negative and the local energy flow vector is non-spacelike (Hawking and Ellis (1973) p.91). Since $d\rho/dp$ can be interpreted as the inverse square of the (adiabatic) velocity of sound in the Newtonian limit we assume also that $d\rho/dp\geq 0$.

We are only interested in non-singular solutions so equation (2.11) and the corresponding equation obtained from the stationary hydrodynamic equilibrium equation (1.11),

$$\log\left(T/T_b\right)=\int_0^p\left(\rho(\bar{p})+\bar{p}\right)^{-1}d\bar{p}, \qquad (2.12)$$

imply that the integral on the right hand side must be finite for all $p\in[0,p_c]$ which is true iff $\lim\limits_{p\longrightarrow 0}p/\rho=0$, i.e. the highly relativistic limit $\rho=p$ cannot occur at extremely low densities, as one expects. This implies that there exists ε, d with $0\leq\varepsilon<1$,

$0<d<\infty$ such that $\lim_{p\to 0} \rho\, p^{-\varepsilon}=d$.

Let us first look at situations in which the matter distribution is not compact. We do this for the spherically symmetric case as we are always interested in solutions close to it. Assume there is no compact boundary hypersurface in Σ given by $p=0$ so that p has an asymptotic expansion $p=O(r^{-k})$ at ∞. Using the asymptotic boundary conditions (1.33) and the equation of hydrostatic equilibrium (1.32) we find that $M=O(r^{-\varepsilon/(1-\varepsilon)})$ at ∞. Together with Einstein's equation $\Delta U=M$ this implies that $\varepsilon\geq 4/5$. (If $\varepsilon=\ell/(\ell+1)$ with $\ell\geq 4$ then $p=(mdr^{-1}(\ell+1)^{-1})^{\ell+1}+O(r^{-\ell-2})$ at infinity.) Since the physically reasonable stellar models can be considered to have compact matter distributions and since in our proofs we use the exact solutions for vacuum obtained in the previous section, we will have to make the assumption that equations of state such that $\varepsilon\geq 4/5$ must be of a form which results in a compact matter distribution.

Now let us look at the star boundary conditions, where $p=0$. If $\varepsilon=0$ $\rho(0)>0$ so ρ is discontinuous at the star boundary. (We will always consider ρ and p to be identically zero in the vacuum region.) In order to investigate the junction conditions further let us assume that $(\ell-1)/\ell<\varepsilon\leq\ell/(\ell+1)$ for some $\ell\geq 0$ (noting that since $\varepsilon\geq 0$ $\varepsilon=0$ if $\ell=0$) and that $\rho(p)$ is C^{ℓ} piecewise $C^{\ell+1}$ on $(0,p_c]$ when ε is in this range. Since U has only one critical point and even in the stationary case we remain close to the spherical solution T also has only one critical point in its domain which is the interior region. From (2.12) we see that in the stationary case p is determined as a function of T since we consider T_b given and fixed. Thus the continuity of derivatives of the mass density across the boundary of the star can be investigated using U (T) in the static (stationary) case to calculate the limit of these derivatives as the star boundary is approached from the interior. If we assumed as much differentiability for $\rho(p)$ as being piecewise C^1 on $[0,p_c]$ then the right hand side of the equation $dp/d\log T=\rho(p)+p$ would be Lipschitz in p on closed intervals of $[0,p_c]$ and a standard existence and uniqueness theorem for first order differential equations (c.f. Hartman (1973)) would imply that $\rho(0)>0$ ($\varepsilon=0$) in order for p not to vanish in the interior region. Thus when $\rho(0)=0$ we must always have that $\lim_{p\to 0} d\rho/dp=\infty$, which is the reason for the differentiability being specified only on $(0,p_c]$. The amount of differentiability demanded in this range is purely for mathematical reasons; in order to have the star boundary conditions determine the proper weighted Sobolev space there cannot be less differentiability in the interior region than there is at the star boundary. This restriction would be very difficult to remove in our formalism but it is not a very serious one as all equations of state could be approximated arbitrarily closely by a C^∞ equation of state.

Let us also assume that $p^{-\varepsilon}\rho$ is C^ℓ in $[0,\delta)$ for some small δ so that

$$\lim_{p\to 0}(d^i\rho/dp^i)p^{i-\varepsilon}=d\varepsilon\cdots(\varepsilon-i+1) \quad \text{for } 0\le i\le \ell.$$

We look first at the junction conditions in the static case. Let z be a C^∞ coordinate defined in a neighborhood of the star boundary such that $z=1$ corresponds to the $p=0$ hypersurface and $p(z)>0$ for $z<1$. The differentiability of m (and ρ and p) can then be investigated using the relation

$$d^\ell M/dz^\ell=\sum_{s=1}^{\ell}\ \sum_{m_1+\cdots+m_s+s=\ell}\ell!\Big(m_1!\cdots m_s!\prod_{i=1}^{s}(i+m_{s-i+1})\Big)^{-1}$$
$$\cdot\, d^sM/dU^s\big(((dU/dz)d/dz)^{m_1}dU/dz\big)\cdots\big(((dU/dz)d/dz)^{m_s}dU/dz\big)$$

This, together with the following argument, will show that the limit as $z\to 1^-$ of $d^\ell M/dz^\ell$ is given, for $(\ell-1)/\ell<\varepsilon\le\ell/(\ell+1)$ by $(dU/dz|_{z=0})^\ell$ times the limit as $p\to 0^+$ of the following expression.

$$d^\ell M/dU^\ell=(-1)^\ell\tfrac{1}{2}e^{-2U}\sum_{i+j=\ell}2^i\big[((\rho+p)d/dp)^j(\rho+p)+2((\rho+p)d/dp)^{j-1}(\rho+p)\big]$$

A long calculation yields

$$\lim_{p\to 0}p^{i-(i+1)\varepsilon}(d^iM/dU^i)=(-1)^i\tfrac{1}{2}e^{-2U_b}d^{i+1}\varepsilon\cdots(i\varepsilon-(i-1))$$

for $0 \le i \le \ell$ and $(\ell-1)/\ell < \varepsilon \le \ell/(\ell+1)$. Recall that $\lim_{z \to 0^+} d^k M/dz^k = 0$ for all k since $M=0$ in the exterior region. Thus M (and ρ) are $C^{\ell-1}$ (piecewise C^ℓ if $\varepsilon = \ell/(\ell+1)$) across the star boundary. We are justified in using U as a coordinate like this since we know it is at least C^1 so we can calculate the first derivative of M and find that M is C^0 (if $\ell > 0$). Einstein's equations then imply that U can then be taken to be C^2 and γ to be C^3 so we can take the second derivative of M, and so on. Thus for $(\ell-1)/\ell < \varepsilon \le \ell/(\ell+1)$ U is $C^{\ell+1}$, γ is $C^{\ell+2}$ and if $\varepsilon = \ell/(\ell+1)$ then U is piecewise $C^{\ell+2}$ and γ is piecewise $C^{\ell+3}$. The equation of hydrostatic equilibrium shows that p is as differentiable as U.

In the stationary case a completely analogous derivation may be done using $\log T$ in place of U, resulting in, for $0 \le i \le \ell$, $(\ell-1)/\ell < \varepsilon \le \ell/(\ell+1)$,

$$\lim_{p \to 0} p^{(i+1)\varepsilon - i} e^{2U} d^i M/d(\log T)^i = \tfrac{1}{2} d^{i+1} \varepsilon \cdots (i\varepsilon - (i-1))$$

Using Einstein's equations (1.22–1.24) the same type of argument shows that U and γ have the same differentiability as in the static case and h is C^ℓ, piecewise $C^{\ell+1}$ if $\varepsilon = \ell/(\ell+1)$.

Our assumptions on the equation of state are then,

(i) $\rho(p) \ge 0$ on some interval $I = [0, p_c]$,

(ii) $d\rho/dp \ge 0$ on I,

(iii) the solution is nonsingular, i.e. there exists ε, d with

$0 \leq \varepsilon < 1$, $0 < d < \infty$ such that $\lim_{p \to 0} p^{-\varepsilon} \rho = d$.

(iv) For $(\ell-1)/\ell < \varepsilon \leq \ell/(\ell+1)$ with $\varepsilon=0$ if $\ell=0$, $\rho(p)$ is C^{ℓ} piecewise $C^{\ell+1}$ on $(0,p_c]$ and $p^{-\varepsilon}\rho$ is C^{ℓ} in $[0,\delta)$ for some small $\delta>0$,

(v) if $\varepsilon \geq 4/5$ the equation of state is such that the finite mass spherical solution with this equation of state has a compact matter distribution.

The results obtained in this section are summarized in the following theorem. The case $\varepsilon=\ell=0$ was given in Künzle and Savage (1980b).

Theorem 2.1: If the equation of state satisfies (i) to (iv) above then the corresponding stellar model consists of a compact region $D \subset \Sigma$ with $\rho \geq p \geq 0$ in D and an exterior vacuum region. For all compact matter distributions the differentiability of M (and ρ) across the boundary ∂D of D, defined by $p=0$, is determined by ε and, for $(\ell-1)/\ell < \varepsilon \leq \ell/(\ell+1)$ with $\ell>0$ M is $C^{\ell-1}$ (and piecewise C^{ℓ} if $\varepsilon=\ell/(\ell+1)$) while M is piecewise C^0 if $\varepsilon=\ell=0$, with the ℓ^{th} derivative normal to ∂D going like const.$p^{(\ell+1)\varepsilon-\ell}$ as $p \to 0$.

(We are now including equations of state of the form $\rho=Ap^q$ for $0<q<1$, the finite stellar models in which ρ tends to 0 on the boundary, which were excluded for mathematical convenience in Künzle and Savage (1980b).)

Let us then denote by $S_{\rho(p)}$ ($\tilde{S}_{\rho(p)}$), or, for short S_ρ ($\tilde{S}_\rho$), the set of static (stationary, axisymmetric) solutions of Einstein's equation with a fixed equation of state $\rho(p)$ satisfying conditions (i) to (v) above, with a fixed surface temperature T_b and such that U has only one critical point. Let $S_{\rho,m}$ ($\tilde{S}_{\rho,m}$) denote the corresponding restriction to solutions with a given constant mass m. Eventually one would like to prove that S_ρ consists only of the spherically symmetric solutions and that $\tilde{S}_{\rho,m}$ consists of a family of solutions which close to the spherical solution is parameterized by the angular momentum J (and for larger J perhaps bifurcate in a similar manner as the Maclaurin– Jacobi–ellipsoid sequence in Newtonian theory). As we have already noted we do not know a priori how large these sets are or what kind of topology and differentiable structure they can be given, but they can be regarded as the inverse image of a differentiable map on a bigger set that can be provided with a fairly natural Banach manifold structure.

Unfortunately there is considerable arbitrariness in the choice of the Banach manifold structure for a set of tensor fields on a non-compact manifold. We try to make the most appropriate choice among the weighted Sobolev spaces which are presented along with some general Banach manifold theory in the next chapter. In chapters 4 and 5 we will see that physical and mathematical considerations combine to limit this choice remarkably. However one cannot exclude the possibility that a manifold of different data and field equations in a different form might perhaps lead to stronger results than we get.

CHAPTER III

MATHEMATICAL PRELIMINARIES

3.1 Differential geometry in relativity

The use of differential topology and differential geometry techniques in relativity theory is certainly not new. For example, the incompleteness theorems of Hawking and Penrose from which the existence of black holes may be inferred—under reasonable mathematical assumptions (c.f. Hawking and Ellis (1973))— and the geometric analysis of spatial and null infinity by conformal mappings use techniques from the study of the topology and geometry of finite dimensional manifolds.

Wheeler (1962) was the first to point out the relevance of infinite dimensional manifold theory to relativity with the introduction of superspace S, the quotient space of riemannian metrics on a given three dimensional manifold obtained by identifying metrics which can be obtained one from another by a coordinate transformation. (S is a metric space but does not have a manifold structure, c.f. Fischer (1970).) The universe can then be viewed as an evolving geometry and thus as a curve in S, allowing a dynamical theory of relativity. This application is regarded as "soft" in that infinite dimensional manifolds are involved mostly as a language convenience and as a guide to the theory's structure. Brill and Deser (1968) gave the first substantial "hard" theorem using infinite dimensional analysis, albeit in an informal way, when they showed that any non-trivial perturbation of Minkowski space leads to a spacetime with strictly positive total gravitational mass. Their technique is to show that on the space of solutions to Einstein's equations the mass function has a non-degenerate critical point at Minkowski space. Later investigations of the positive mass conjecture have used both infinite dimensional manifold theory (Choquet-Bruhat et al. (1979)) and finite dimensional manifold theory (Schoen and Yau (1979) and Jang (1979)).

Our approach also uses infinite dimensional manifold theory and is based on the problem of linearization stability which has been investigated by Fischer and Marsden (1975a, 1975b). The reason for this is as follows, where we will restrict to the static case as the reasoning is the same in the stationary case. We can regard $S_{\rho,m}$ as the inverse image of 0 of a differentiable map $\mathscr{L}:X\longrightarrow Y$, obtained from Einstein's equations, between suitable Banach spaces X and Y where a point $\sigma\in X$ will uniquely specify a spacetime. We also know that there is a unique spherically symmetric solution $\sigma\in\mathscr{L}^{-1}(0)$ (by including a specification of U_b in the space X if necessary) and we want to answer the question what does the solution set $\mathscr{L}^{-1}(0)$ look like? To answer this question globally is a formidable problem and has been done only in the special case where ρ=const. by a similar approach to that taken in the black hole theorems

(Lindblom (1980)), as we have noted. In order to have a more general approach (and the generality is proven by its applicability to the stationary case as well) we attempt to answer the question locally, near σ, by considering solutions in $S_{\rho,m}$ which are close to σ in X. However $\mathscr{L}$ is formed from Einstein's equations and is thus highly non-linear so a direct approach is unfeasible. The usual approach in such situations is to linearize the equations, solve these linearized equations and assert that they are an approximation to the true solution of the non-linear equations. More explicitly, for σ near σ write $\sigma(\lambda)$ for a parameter λ and expand as $\sigma(\lambda)=\sigma+\lambda\sigma_1+\lambda^2\sigma_2+\cdots$. The approximation to first order is $\sigma+\lambda\sigma_1$ where $\sigma_1=(d\sigma/d\lambda)|_{\lambda=0}$. Demanding that $\sigma(\lambda)\in\mathscr{L}^{-1}(0)$, i.e. $\mathscr{L}(\sigma(\lambda))=0$, we find that $\mathscr{L}'(\sigma)\sigma_1=0$ where $\mathscr{L}'(\sigma):T_\sigma X\longrightarrow T_{\mathscr{L}(\sigma)}Y$ is the tangent map.

The implicit assumption that the solution to the linearized equations is an approximation to the full equations is, however, not always valid. In fact Fischer and Marsden (1975a) have shown that if the universe is toroidal, $T^3\times\mathbb{R}$, where T^3 denotes the flat 3-torus, this assumption is not valid for the flat space Einstein equations $R_{\nu\mu}(g)=0$. For a more visible example of such a situation consider $f:\mathbb{R}^2\longrightarrow\mathbb{R}$, $(x,y)\longmapsto x^2-y^2$. Then $f^{-1}(0)$ $=\{(x,y)|y=\pm x\}$ but $f'(0,0)h=0$ for all $h=(h_1,h_2)\in T_{(0,0)}\mathbb{R}^2=\mathbb{R}^2$. Only by going to the second order condition $f''(0,0)(h,h)=2(h_1^2-h_2^2)=0$ $\Longleftrightarrow h_2=\pm h_1$ do we get a true approximation. Situations in which this assumption is valid will be called linearization stable.

Definition 3.1: Let E and F be topological vector spaces and $L:E \longrightarrow F$ a differentiable mapping. We say L is linearization stable at $x_0 \in E$ iff for every $h \in E$ such that $L'(x_0)h=0$ there exists a differentiable curve $x(t) \in E$ with $x(0)=x_0$, $L(x(t))=L(x_0)$ and $x'(0)(:=dx/dt|_{t=0})=h$.

The vacuum Einstein equations have been well studied in this manner by considering the induced metric $\underset{\sim}{g}$ and the second fundamental form $\tilde{\Omega}$ on a spacelike hypersurface of the spacetime as initial data, which satisfy some nonlinear constraint equations, for some evolution equations. By using various elliptic operator techniques Fischer and Marsden (1975a) have proven the following theorem.

Theorem 3.1: Suppose there is such a spacelike hypersurface N of the spacetime with $\underset{\sim}{g}$ and $\tilde{\Omega}$ satisfying (i) there are no infinitesimal isometries η on both $\underset{\sim}{g}$ and $\tilde{\Omega}$ (with η vanishing at infinity if N is not compact), (ii) if $\tilde{\Omega}=0$ and N is compact then $\underset{\sim}{g}$ is not flat, (iii) if $\tilde{\Omega}\neq 0$, $tr(\tilde{\Omega})=$ trace of $\tilde{\Omega}$ is constant on N if N is compact and $tr(\tilde{\Omega})=0$ if N is noncompact, (iv) if N is noncompact, $\underset{\sim}{g}$ is complete and asymptotically flat. Then near N, $R_{\mu\nu}(g)=0$ is linearization stable.

(See Choquet-Bruhat et al. (1977a) for a more recent, simpler proof based on some results of weighted Sobolev spaces.) The toroidal example fails because condition (ii) fails for $N=T^3$. The corollary that in Minkowski space the vacuum Einstein equations are linearization stable was obtained independently by Choquet-Bruhat and Deser (1973). (In this case the linearized equations are the weak field approximations used to study gravitational waves.)

We will be following the spirit and not the details of this work on linearization stability. In fact, as we will see, our tangent map $\dot{\mathcal{L}}'$ is not surjective so we cannot show that our equations are linearization stable. However, the differential structure which we choose for the Banach space X will allow a slicing for the action of the diffeomorphism group on X near σ which together with some theorems on elliptic operators enables us to prove a theorem about the physically distinct solutions in $\mathcal{L}^{-1}(0)$ near σ. X must clearly be an infinite dimensional space so in the next section we take a look at infinite dimensional Banach spaces. The following section will then present the weighted Sobolev spaces introduced by Cantor which we use to model X on as well as two theorems due to Cantor (1979) and Nirenberg and Walker (1973) for elliptic operators which are "differentiably close" to elliptic operators with constant coefficients which we use to extend our analysis of $\mathcal{L}'(\sigma)$ to a neighborhood of σ in X. It is these latter two theorems which together with the slicing theorem allow us to prove our results about curves of physically distinct solutions, even though we have not shown linearization stability (so that we cannot prove existence).

3.2 Banach spaces and infinite dimensional manifolds

A Banach space E is a complete normed vector space, while a Banach manifold N is a (Hausdorff) topological space with a maximal atlas of charts $\{(\varphi_\alpha, U_\alpha)\}$ such that each φ_α: $U_\alpha \longrightarrow V_\alpha \subset E$ where V_α is open in E, is a homeomorphism. In particular a Banach space or an open subset of a Banach space is a Banach manifold. $S \subset N$ is a submanifold of N if we can write $E = F \oplus G$ (topological sum) and for every $x \in S$ there is a chart $\varphi : U \subset N \longrightarrow V \subset E$ of N where $x \in U$ such that $\varphi(U \cap S) = V \cap (F \times \{w\})$ where $w \in G$. In other words, the chart φ "flattens out" S making it lie in the subspace F. When a closed subspace F of E is such that $E = F \oplus G$ where G is another closed subspace of E, we say that F splits (c.f. Marsden (1974)).

Such splittings clearly always occur in the case where E is finite dimensional or a Hilbert space (where the inner product can be used to find the orthogonal complement to F). However, "intuition" from finite dimensional spaces sometimes fails in infinite dimensional spaces. For example, in $\mathbb{R}^n$ a closed, bounded subset with non-empty interior is always compact whereas in an infinite dimensional Banach space it is never compact under the norm topology. More importantly for our work, as we will soon see, an arbitrary closed subspace F of an infinite dimensional Banach space does not always split. Fortunately, though, the generalization from differential calculus on $\mathbb{R}^n$ to differential calculus on Banach spaces is remarkably smooth with many theorems, including the inverse and implicit function theorems, similarly phrased (c.f. Choquet-Bruhat et al. (1977b)). These latter two theorems are important for obtaining many substantial results in Banach space theory, some of which are the following.

Let N and R be Banach manifolds and $f: N \longrightarrow R$ a C^1 map. We are interested in when $S=f^{-1}(y_0)$ is a submanifold of N and when f is linearization stable on S (i.e. we would like $\mathscr{L}^{-1}(0) \subset X$ to be a submanifold with its tangent space at σ determined by $\mathscr{L}'(\sigma)=0$ for σ in a neighborhood of σ). Suitable conditions are given by the following theorem (c.f. Fischer and Marsden (1975a)).

> Theorem 3.2: Let $x_0 \in N$ and $f(x_0)=y_0$. Suppose that $f'(x_0)$ is surjective and that kernel $f'(x_0)$ splits. Then $f^{-1}(y_o)$ is a C^1 submanifold near x_0 and $f(x)=y_0$ is linearization stable about x_0.

Proof: Work in a chart $U \subset E$ for N and write $E=E_1 \oplus E_2$ where $E_1=\ker f'(x_0)$. Consider the map Φ, defined near x_0 to $E_1 \times F$ (where R is modelled on the Banach space F) by $\Phi(x_1,x_2) =(x_1,f(x_1,x_2))$. Since $(f|_{\{x_1\}\times E_2})'(x_0)$ is an isomorphism, $\Phi'(x_0)$ is an isomorphism so the inverse function theorem implies that Φ is a local diffeomorphism. Thus Φ^{-1} gives a chart for $f^{-1}(y_0)$ near x_0 modelled on E_1 and $T_{x_0}f^{-1}(y_0)=\ker f'(x_0)$. Now $h \in T_{x_0}N$ is a first order deformation iff $h \in \ker f'(x_0)$ and since $f^{-1}(y_0)$ is a submanifold, there exists a curve $x(\lambda) \in f^{-1}(y_0)$ which is actually tangent to h. Thus f is linearization stable about x_0. ■

There are thus two properties of $\mathscr{L}'(\sigma): T_\sigma X \longrightarrow T_{\mathscr{L}(\sigma)}Y$ in which we are interested, surjectivity and the splitting of its kernel. Clearly surjectivity depends on the image space Y —it must be large enough but not too large. We will see later that in our case it is too large but it is by no means obvious how to pick a submanifold of Y which will be the right size. The splitting of the kernel is not as formidable. In fact much work has been done on orthogonal decompositions of symmetric covariant tensor fields S_2 (of which the metrics are an open subset) over both compact and noncompact manifolds (Berger and Ebin (1969), York (1974), Cantor (1979)). In the case of a noncompact manifold though, the decomposition must have some dependence on the space S_2, such as its asymptotic properties. In the next section we will see see some spaces which are suitable for modelling the asymptotically flat metrics on Σ, but there will be a problem with the rate of falloff at infinity required for such a splitting. This can fortunately be resolved.

But why are we interested in this splitting property if we can not show linearization stability anyway? The answer is that this sort of splitting together with some properties of weighted

Sobolev spaces (and in the stationary case another splitting theorem for vector fields as well) enables one to show that near σ the orbits due to the action of the diffeomorphism group are "stacked" in the sense that in a neighborhood V of σ there is a submanifold $\mathcal{S}$, called a slice, which passes through σ and such that all the orbits in this neighborhood V pass through σ only once. Thus $\mathcal{S}$ represents the physically distinct spacetimes which are close to σ. Thus given any "physical" curve of solutions in $\mathcal{L}^{-1}(0)\cap\mathcal{S}$ the behaviour of the tangent to such a curve near σ can be investigated with the use of the theorems about elliptic operators. By choosing the space X well, whether there is such a nonconstant curve will tell us how unique the spherically symmetric solution is, at least locally, in a similar manner to that of the stationary black hole theorem.

Before investigating spaces to model X on, we state a similar result to that of the submersion result of theorem 3.2 which we use in some later proofs.

Theorem 3.3: Let $f:N\longrightarrow R$ be injective and closed. If $f'(x):T_xN\longrightarrow T_{f(x)}R$ is injective and its image splits for each $x\in N$ then $f(N)$ is a submanifold of R.

Proof: (c.f. Lang (1972) p.27)

3.3 Weighted Sobolev spaces

The definitions and theorems (except theorem 3.14) in this section come from Cantor's (1975a,1975b,1979) work on the *weighted Sobolev spaces* $M^p_{s,\delta}$ which he introduced. (The relevant norm was suggested by some results of Nirenberg and Walker (1973).) As will become readily apparent, these spaces allow both the asymptotic properties and the junction conditions at the star boundary to be readily incorporated into our argument. As noted in §2.2 this is clearly a necessary requirement for any uniqueness argument.

Let $p \ge 0$ and let $\| \ \|_p$ denote the L^p-norm on the set $C^\infty_0(\mathbb{R}^n,\mathbb{R}^m)$ of C^∞ maps with compact support. Then $M^p_{s,\delta} = M^p_{s,\delta}(\mathbb{R}^n,\mathbb{R}^m)$ is defined to be the completion of C^∞_0 with respect to the norm

$$\|f\|_{p,s,\delta} := \sum_{|\mu|\le s} \|\sigma(x)^{(|\mu|+\delta)} D^\mu f\|_p \quad (s\in\mathbb{N},\delta\in\mathbb{R}) \tag{3.1}$$

where $\sigma^2(x) := 1+|x|^2$, $\mu=(\mu_1,\cdots,\mu_n)\in\mathbb{N}^n$, $|\mu| := \Sigma\mu_i$ and $D^\mu f := \partial^{|\mu|} f/(\partial x^{\mu_1}\cdots\partial x^{\mu_n})$. If $f\in C^\infty(\mathbb{R}^n,\mathbb{R}^m)$ then define also

$$M^p_{s,\delta}(f) := \{h:\mathbb{R}^n\longrightarrow\mathbb{R}^m \mid (h-f)\in M^p_{s,\delta}\}. \tag{3.2}$$

The $M^p_{s,0}$ spaces are not just the usual *Sobolev spaces* $W^{s,p}$ which are defined to be the completion of C^∞_0 with respect to the norm

$$\|f\|_{p,s} = \sum_{|\mu|\le s} \|D^\mu f\|_p. \tag{3.3}$$

Cantor originally introduced the weighted Sobolev spaces $M^p_{s,\delta}$ in order to apply the techniques of global nonlinear analysis, such as that used by Fischer and Marsden, to physical problems over unbounded regions in space where the boundary conditions include the asymptotic behaviour of the solutions as $|x|\longrightarrow\infty$, such as classical fluid mechanics with steady flow at infinity, or isolated relativ~~istic fluids~~. ~~This is done~~ by ~~specifying an~~ $M^p_{s,\delta}$ space of solutions which satisfies the desired asymptotic conditions and studying the behaviour of the particular differential operators given by the physical problem on these spaces, which is what we want to do. For bounded regions and compact manifolds the Sobolev spaces $W^{s,p}$ are commonly used (c.f. Ebin and Marsden (1970)). However the properties of a differential operator can be very different on the different spaces. For example the Laplace operator from $W^{s,p}$ to $W^{s-2,p}$ is not surjective but, as we will see, it is surjective from $M^p_{s,\delta}$ to $M^p_{s-2,\delta+2}$ for certain p, s and δ.

The asymptotic behaviour can be seen by letting f admit an asymptotic expansion $f(x)=|x|^{-\nu}\sum_{k=0}^{\infty}(|x|^{-k}f^k(x))$ with $x^i\partial_i f^k=0$, $f^0\neq 0$. Then $f\in M^p_{s,\delta}$ iff $\nu>\delta+n/p$. Any $f\in M^p_{s,\delta}$ which does not admit an asymptotic expansion can clearly stay "large" as $|x|\longrightarrow\infty$ on an angular portion $\Delta\Omega(x)$ which goes to zero as $|x|\longrightarrow\infty$ or on annular rings whose width approaches zero as $|x|\longrightarrow\infty$. If f is C^0 however neither of these situations can arise.

The inclusion maps

$$M^p_{s_1,\delta} \longrightarrow M^p_{s_2,\delta} \text{ for } s_1 \geq s_2 \text{ and } M^p_{s,\delta_1} \longrightarrow M^p_{s,\delta_2} \text{ for } \delta_1 \geq \delta_2 \tag{3.4}$$

are easily seen to be continuous. The Sobolev imbedding theorem, which is one of the main hard theorems of Sobolev space theory and states that $W^{s,p} \longrightarrow C^k$ continuously for $s > k + n/p$ (c.f. Adams (1975)), shows that

$$M^p_{s,\delta} \longrightarrow C^k \text{ is continuous if } \delta \geq 0 \text{ and } k + n/p < s \tag{3.5}$$

since $\|f\|_{p,s,\delta} \leq \|f\|_{p,s}$ for any $f \in M^p_{s,\delta}$. Recall that C^k is the completion of C^∞_0 with respect to the norm

$$\|f\|_{C^k} = \max_{0 \leq |\mu| \leq k} \sup_{x \in \mathbb{R}^n} |D^\mu f(x)|.$$

Note that a C^{k-1} and piecewise C^k function may still be in $M^p_{s,\delta}$ for $s \leq k$ (if its asymptotic properties are right) but not in $M^p_{k+1,\delta}$.

It is easy to see that partial differentiation induces a continuous map

$$\partial_k : M^p_{s,\delta} \longrightarrow M^p_{s-1,\delta+1} \tag{3.6}$$

which is also linear and thus C^∞.

If $p > 1$, $s > n/p$, $0 \leq k \leq s$, $\delta, \delta' \geq 0$ then any pointwise multiplication $\mathbb{R}^m \times \mathbb{R}^{m'} \longrightarrow \mathbb{R}^{m''}$ induces continuous maps

$$M^p_{s,\delta}(\mathbb{R}^n,\mathbb{R}^m) \otimes M^p_{s-k,\delta+k}(\mathbb{R}^n,\mathbb{R}^{m'}) \longrightarrow M^p_{s-k,\delta+k}(\mathbb{R}^n,\mathbb{R}^{m''}), \tag{3.6}$$

$$M^p_{s,\delta}(\mathbb{R}^n,\mathbb{R}^m) \otimes M^p_{s,\delta'}(\mathbb{R}^n,\mathbb{R}^{m'}) \longrightarrow M^p_{s,\delta+\delta'}(\mathbb{R}^n,\mathbb{R}^{m''}). \tag{3.7}$$

This is sometimes referred to as the Schauder ring property. Let us prove (3.6). It suffices to show that $(f,g) \longrightarrow D^\mu(fg)$ is continuous from $M^p_{s,\delta} \otimes M^p_{s-k,\delta+k} \longrightarrow M^p_{0,\delta+k+|\mu|}$ for all $|\mu| \leq s-k$. Using Leibnitz' rule one gets

$$\|\sigma^{\delta+k+|\mu|} D^\mu(fg)\|_p \leq C_1 \sum_{|\nu| \leq |\mu|} \|(\sigma^{|\nu|} D^\nu f)(\sigma^{\delta+k+|\mu-\nu|} D^{\mu-\nu} g)\|_p$$

for some positive constant C_1. It is known that if $q+t>n/p$ then pointwise multiplication is continuous from $W^{q,p} \otimes W^{t,p} \longrightarrow L^p$ (c.f. Adams (1975)). It is easily seen that $D^\nu f \in M^p_{s-|\nu|,\delta+|\nu|}$ and that $\sigma^{\delta+k+|\mu-\nu|} D^{\mu-\nu} g \in W^{s-k-|\mu-\nu|,p}$ so since $s-|\nu|+s-k-|\mu-\nu| \geq s > n/p$ there is a constant C_2 such that

$$\|\sigma^{\delta+k+|\mu|} D^\mu(fg)\|_p \leq C_2 \|f\|_{p,s,\delta} \|g\|_{p,s-k,\delta+k}.$$

Before proceeding to more theorems about these spaces, perhaps we should pause to note the relation between the $M^p_{s,\delta}$ spaces when the underlying space is a Riemannian manifold $N=(\mathbb{R}^n,g)$ where g is complete instead of a Euclidean space $E^n=(\mathbb{R}^n,e)$ where e is the Euclidean metric and to note that these theorems can be applied to appropriate tensor fields. Letting $\| \, \|$ be the norm generated by g on a tensor space $T^r_q N$ and dV be the volume element generated by g we can define $M^p_{s,\delta}(T^r_q N)$, in a natural way, to be the completion of $C^\infty_0(T^r_q N)$ with respect to the norm

$$\|T\|_{p,s,\delta} = \sum_{|\mu|\le s} \left(\int \|\sigma^{|\mu|+\delta} \nabla^\mu T\|^p dV \right)^{1/p} \tag{3.9}$$

for $p\ge 1$, $s\in\mathbb{N}$, $\delta\in\mathbb{R}$. (Similar definitions can be made for $W^{s,p}$ and C^k tensor fields.) Replacing $\sigma(x)$ with $(1+d(x,0)^2)^{1/2}$ where d is the distance function generated by g gives an equivalent norm under suitable assumptions on g. In fact, from Cantor (1979) we have the following.

Lemma 3.4: When N is such that

$\lim_{|x|\to\infty} \sup|\sigma^{|\mu|}(x) D^\mu (g-e)(x)|=0$ for $|\mu|\le m$ where $m\ge \max(k,s+2)$, the $W^{s,p}$, C^k and $M^p_{s,\delta}$ norms generated by g and those generated by e are equivalent.

This allows one to treat e as a "background" metric for N. Cantor (1979) uses this to transfer results obtained about the diffeomorphism group on flat space (Cantor (1975b)) to results about the diffeomorphism group on N. We will simply state the relevant definitions and theorems.

Let

$$\mathcal{D}^p_{s,\delta} = \{\varphi\in M^p_{s,\delta}(I) | \varphi^{-1} \text{ exists and } \varphi^{-1}\in M^p_{s,\delta}(I)\} \tag{3.10}$$

where I is the identity map on $\mathbb{R}^n$. This is the set of diffeomorphisms asymptotic to the identity at infinity. Such diffeomorphisms will not destroy the asymptotic behaviour of tensor fields provided they fall off fast enough as can be seen from the following theorem.

Theorem 3.5: Let $p>1$, $s>1+n/p$, $\delta\geq 0$. Then $\mathcal{D}^p_{s,\delta}(N,N)$ is a topological group under compositon (right composition is smooth) and a smooth Banach manifold (in fact it is an open subset of $M^p_{s,\delta}(I)$). For $\delta_1\geq\delta$ composition from $M^p_{k,\delta_1}\oplus\mathcal{D}^p_{s,\delta}\longrightarrow M^p_{k,\delta_1}$ is continuous for all $k\leq s$ and if $f:\mathbb{R}^n\longrightarrow\mathbb{R}^m$ is such that $\sigma^{|\mu|+\delta_1-1}D^\mu f$ is a bounded map for all $|\mu|\leq s$ then composition from $M^p_{k,\delta_1}(f)\oplus\mathcal{D}^p_{s,\delta}\longrightarrow M^p_{k,\delta_1}$ is continuous for all $k\leq s$.

Proof: This corresponds to theorems 1.2, 1.6, 1.7 and 1.9 of Cantor (1975b) and theorem 2.10 of Cantor (1979).

For convenience, let us denote both $M^p_{s,\delta}$ vector fields and $M^p_{s,\delta}$ 1-forms by $X^p_{s,\delta}$ and $M^p_{s,\delta}$ symmetric covariant 2-tensors by $S^p_{s,\delta}$. Since specifying a spacetime involves specifying a positive definite, asymptotically flat metric γ on $\Sigma\cong\mathbb{R}^3$ we are clearly going to be interested in the set

$$\mathcal{R}^p_{s,\delta}=\{g\,|\,g-e\in S^p_{s,\delta}\text{ and }g\text{ is positive definite}\},$$

the set of Riemannian metrics which are asymptotic to the identity. $\mathcal{R}^p_{s,\delta}$ has a natural Banach space structure for $s>n/p$, being an open cone in $S^p_{s,\delta}(e)$.

Since the set of metrics $g\in\mathcal{R}^p_{s,\delta}$ which satisfy the conditions of lemma 3.4 for $N=(\mathbb{R}^n,g)$ is dense and on this dense subset $\mathcal{D}^p_{s,\delta}(N,N)$ does not depend on g, from now on we will consider $\mathcal{D}^p_{s,\delta}$ to denote $\mathcal{D}^p_{s,\delta}(E^n,E^n)$ (i.e. the set of diffeomorphisms near the identity with respect to e).

If we are to gain any insight into the physically distinct solutions which correspond to the solutions of the linearized equation $\mathcal{L}'(\sigma):T_\sigma X\longrightarrow T_{\mathcal{L}(\sigma)}Y$ we must have some way of factoring out the infinite dimensional space of solutions corresponding to the coordinate freedom on Σ. Recalling that a positive definite metric γ on Σ is part of the specification of a spacetime $\sigma\in X$ it is clear that the action of the diffeomorphisms on $\mathcal{R}^p_{s,\delta}$ and their resultant orbits are going to determine the action and corresponding orbits for X. With this in mind we will now present the relevant theorems dealing with the action of the diffeomorphism group on $\mathcal{R}^p_{s,\delta}$, beginning with the following theorem of Cantor (1979).

Theorem 3.6: Let $p>1$, $s>1+n/p$ and $\delta\geq 0$. Then $\mathcal{D}^p_{s,\delta}$ has a continuous action on $\mathcal{R}^p_{s-1,\delta+1}$ given by

$$A:\mathcal{D}^p_{s,\delta}\times\mathcal{R}^p_{s-1,\delta+1}\longrightarrow\mathcal{R}^p_{s-1,\delta+1}:(\varphi,g)\longrightarrow\varphi^*(g).$$

Moreover A is a C^∞ function of g and if $g \in R^p_{s-1+k,\delta+1}$ then $\varphi \longrightarrow \varphi^*(g)$ is a C^k function on $\mathcal{D}^p_{s,\delta}$.

Proof: Setting $g=e+h$ with $h \in S^p_{s-1,\delta+1}$ we have $\varphi^*(g)=\varphi^*(e)+\varphi^*(h)$. In coordinates, $\varphi^*(e)_{ij}=(\partial\varphi^k/\partial x^i)(\partial\varphi^l/\partial x^j)\delta_{kl}$ and $\varphi^*(h)_{ij}=(\partial\varphi^k/\partial x^i)(\partial\varphi^l/\partial x^j)(h\circ\varphi)_{kl}$. Using the Schauder ring property together with $\varphi \longrightarrow \partial\varphi$ being smooth from $M^p_{s,\delta}$ into $M^p_{s-1,\delta+1}$ so that $\partial\varphi^r/\partial x^s=\delta^r_s+B^r_s$ with B^r_s in $M^p_{s-1,\delta+1}$, it is seen that $\varphi \longrightarrow \varphi^*(e)$ is smooth. For $h \in M^p_{s-1+k,\delta+1}$ theorem 3.5 implies $(h,\varphi) \longrightarrow h\circ\varphi$ as a function from $M^p_{s-1,\delta+1} \times \mathcal{D}^p_{s,\delta} \longrightarrow M^p_{s-1,\delta+1}$ is C^∞ in h and C^k in φ, completing the proof. ■

How can the coordinate freedom be factored out of $\mathcal{R}^p_{s-1,\delta+1}$? Ideally one would like to deal with the quotient space $\mathcal{R}^p_{s-1,\delta+1}/\mathcal{D}^p_{s,\delta}$, the orbit space of the group of diffeomorphisms acting on the Riemannian metrics, linearizing the equations on the quotient space. But such quotient spaces do not in general have a manifold structure. In fact superspace, the quotient space obtained from the space of C^∞ Riemannian metrics $\mathcal{R}$ and C^∞ diffeomorphisms $\mathcal{D}$ on a compact oriented manifold, is not a manifold (Fischer (1970)). This results because for symmetric metrics—i.e. metrics invariant under some continuous coordinate transformation—possessing a non-trivial isometry group $I_g=\{\varphi\in\mathcal{D} | \varphi^*(g)=g\}$ the appropriate diffeomorphism group to form the quotient space with is $\mathcal{D}/I_g$ since an isometric metric (one of the same functional form as g) generated by the action of a diffeomorphism is not considered to be "new". Thus in the quotient space where the orbits are identified to points the neighborhood of a symmetric geometry will not be homeomorphic to neighborhoods of generic geometries (geometries with no symmetries). However superspace can be viewed as a "stratification" of manifolds of geometries with given symmetries, the more symmetric geometries forming a manifold boundary to a manifold of geometries of less symmetry.

Because we are dealing with diffeomorphisms asymptotic to the identity the problem of symmetries is "pushed out to infinity" and in fact the isometry group $I_g=\{\varphi\in\mathfrak{D}^p_{s,\delta}|\varphi^*(g)=g \text{ for } g\in\mathcal{R}^p_{s-1,\delta+1}\}$ is trivial so the above problem does not arise. However another difficulty resulting from the differentiability conditions and therefore appearing also, as well as the symmetry problem, in the case of $W^{s,p}$ metrics and diffeomorphisms (Ebin (1970)) is present. This is related to the slicing theorems. Roughly speaking, when there are no nontrivial isometries of g a slice at g is a subspace which is "orthogonal" to the orbit through g and which, together with a small neighborhood of the orbit, fills out an open neighborhood of g. When everything is C^∞ the orbits through any g are C^∞ submanifolds and a slice always exists (Ebin (1968)). Fischer

(1970) uses this "stacking" of orbits to show that geometries with the same symmetry form a manifold. In the $W^{s,p}$ or $M^p_{s,\delta}$ spaces of metrics the orbits are submanifolds but are in general only C^0 submanifolds and only at the more differentiable metrics does a slice exist. There thus seems little hope in establishing a useful structure for the quotient space $\mathcal{R}^p_{s-1,\delta+1}/\mathcal{D}^p_{s,\delta}$.

How then can the coordinate freedom be dealt with? It turns out that by carefully choosing the functions with which we define the topology of X we can make the unique spherical solution $\sigma \in X$ have enough differentiability so that a slice $\mathcal{S}$ does exist at σ. Thus although no global statements can be made we can still deal locally with a submanifold of physically distinct spacetimes.

Let us begin the presentation of the appropriate slicing theorems by stating one of the two main theorems about elliptic operators which are "differentiably close" to elliptic operators with constant coefficients that we use in our proofs. We will refer to this theorem as Cantor's isomorphism theorem. Recall first that a differential operator $A = \sum_{|\mu| \le k} a_\mu(x) D^\mu$ on $\mathbb{R}^n$ is an elliptic k^{th} order differential operator if there are no real solutions $\chi \in \mathbb{R}^n$ of $\sum_{|\mu|=k} a_\mu(x) \chi^\mu = 0$ for all $x \in \mathbb{R}^n$.

Theorem 3.7: Let $n > k$ and $A_\infty = \sum_{|\mu|=k} \bar{a}_\mu D^\mu$ a homogeneous elliptic operator with constant coefficients on $\mathbb{R}^n$ and $A(x)=A_\infty+\sum_{|\mu|\leq k} b_\mu(x)D^\mu$ an elliptic operator. Then, if $p>n/(n-k)$, $0\leq\delta<n-k-n/p$, $s\geq k$, $s>n/p$ and $b_\mu\in M^p_{s-k+|\mu|,k-|\mu|}$ with b_μ continuous outside a compact set on $\mathbb{R}^n$ for $|\mu|\leq n/p$, A maps $M^p_{s,\delta}$ continuously into $M^p_{s-k,\delta+k}$ with closed range and finite dimensional kernel. Moreover, suppose that either

(i) $\sum_{|\mu|\leq k}\|b_\mu\|_{p,s-k,k-|\mu|}<\varepsilon$ for sufficiently small ε, or

(ii) There is a continuous curve c from $[0,1]$ into the space of bounded linear operators between $M^p_{s,\delta}$ and $M^p_{s-k,\delta+k}$ such that $c(0)=A_\infty$, $c(1)=A$ and for each $t\in[0,1]$, $c(t)$ is an injection and satisfies the hypothesis of the theorem.

Then A is an isomorphism of $M^p_{s,\delta}$ onto $M^p_{s-k,\delta+k}$.

Proof: Let $f\in M^p_{s,\delta}$. By Leibnitz' rule, for $|\beta|\leq s-k$,

$\|\sigma^{\delta+k+|\beta|}D^\beta(b_\mu D^\mu f)\|_p\leq C_1\sum_{|\nu|\leq|\beta|}\|(\sigma^{k-|\mu|+|\nu|}D^\nu b_\mu)(\sigma^{\delta+|\mu|+|\beta-\nu|}D^{\beta-\nu+\mu}f)\|_p$ and it is readily seen that $D^\nu b_\mu\in M^p_{s-k-|\nu|+|\mu|,k-|\mu|+|\nu|}$ and $D^{\mu+\nu-\beta}f\in M^p_{s+|\nu|-|\beta|-|\mu|,\delta-|\nu|+|\beta|+|\mu|}$. Since $(s-k-|\nu|+|\mu|)+(s+|\nu|-|\beta|-|\mu|)\geq s>n/p$ multiplication from $M^p_{s-k+|\mu|,k-|\mu|}\times M^p_{s-|\mu|,\delta+|\mu|}\longrightarrow M^p_{s-k,\delta+k}$ is continuous and there exists a constant C_2 such that

$$\|b_\mu D^\mu f\|_{p,s-k,\delta+k}\leq C_2\|b_\mu\|_{p,s-k,k-|\mu|}\|f\|_{p,s,\delta}. \qquad (3.11)$$

Thus, as shown in Cantor (1979), if f_R denotes functions whose support is in $\{x|x>R\}$ with $\|f_R\|_{p,s,\delta}=1$ then $\lim_{R\longrightarrow\infty}\|(A-A_\infty)f_R\|_{p,s-k,\delta+k}$ $\leq\sum_{|\mu|\leq k}\lim_{R\longrightarrow\infty}\|b_\mu D^\mu f_R\|_{p,s-k,\delta+k}=0$ and since $b_\mu\in C^{s-k}$ for $|\mu|>n/p$ all the b_μ are continuous outside a compact set so $\lim_{|x|\longrightarrow\infty}\sup|\sigma^{k-|\mu|}b_\mu(x)|=0$. This satisfies the conditions of theorem 1.4 of Cantor (1979), where $b_\mu\in M^p_{s-k,k-|\mu|}$ with $s>k+n/p$ was assumed, the latter condition ensuring that multiplication was continuous and that the b_μ were continuous.

The outline of his proof is as follows. That A has finite dimensional kernel follows directly from theorem 4.1 of Nirenberg and Walker (1973). The closed range result is obtained by showing that for all $f\in M^p_{s,\delta}\setminus\ker A$, $\|f\|_{p,s,\delta}$ $\leq C\|Af\|_{p,s-k,\delta+k}$ for some constant C, using a similar argument to the one we use in the proof of theorem 3.14. The isomorphism results (i) and (ii) follow from lemma 1.1 of Cantor (1979) which shows that $A_\infty:M^p_{s,\delta}\longrightarrow M^p_{s-k,\delta+k}$ is an isomorphism together with (i) the relation (3.11) or (ii) a lemma essentially due to Schauder which states that for two continuous linear maps A_0, A_1 between Banach spaces E and F such that A_0 is an isomorphism and such that there is a continuous curve c from $[0,1]$ into the space of bounded operators from E to F with $c(0)=A_0$, $c(1)=A_1$ and $c(t)$ an injection with closed range for each $t\in[0,1]$, then A_1 is an isomorphism. ■

This theorem is used in three ways. First, we use the closed range portion of the theorem directly, together with theorem 3.14 (essentially due to Nirenberg and Walker) in § 4.4 to extend the results about the kernel of an elliptic inhomogeneous operator which we obtain from $\mathcal{L}'(\sigma)$ to the kernel of the corresponding operator obtained from $\mathcal{L}'(\sigma)$ for σ in a neighborhood of σ. ($\mathcal{L}'(\sigma)$ is not elliptic.) Secondly, the isomorphism property of some specific elliptic operators is used to obtain the vanishing of some tensor fields which satisfy certain homogeneous elliptic equations. The following corollaries are useful in this regard.

Corollary 3.8: Let $\Delta_g = \mathrm{div}_g \circ d$ be the Laplace–Beltrami operator (where $\mathrm{div}_g \xi = -\nabla^i \xi_i$ for a 1-form ξ). If $p > n/(n-2)$, $s \geq 2$, $0 \leq \delta < n-2-n/p$ and $g \in \mathcal{R}^p_{s,\delta}$ then $\Delta_g : M^p_{s,\delta} \longrightarrow M^p_{s-2,\delta+2}$ is a continuous isomorphism. (The subscript g will be dropped when the metric is clear.)

Proof: (Cantor (1979)) That $\Delta_e : M^p_{s,\delta} \longrightarrow M^p_{s-2,\delta+2}$ is an isomorphism was shown by Cantor (1975a) using estimates of Nirenberg and Walker (1973). For $f \in M^p_{s,\delta}$, $\Delta_g f = (\mathrm{div}_g \circ d) f = -|g|^{-1/2} \sum_{i,j} \partial/\partial x^i (|g|^{1/2} g^{ij} \partial f/\partial x^j)$ where $|g| = \det(g_{ij})$. It is easily verified from this together with $g \in \mathcal{R}^p_{s,\delta}$ that Δ_g satisfies the hypothesis of theorem 3.4. Writing $A_\infty = \sum_{i=1}^{n} \partial^2/\partial x^{i^2}$ we find that each $A_t = A_\infty + t(\Delta_g - A_\infty)$ is an elliptic operator with no lowest order term and therefore satisfies the maximum principle (c.f.

Friedman (1969) p.88). In particular, each A_t has no non-trivial bounded solution and so it is an injection on $M^p_{s,\delta}$. Thus condition (ii) of theorem 3.7 holds and Δ_g is an isomorphism.

■

Corollary 3.9: Let $K_g : X^p_{s,\delta} \longrightarrow S^p_{s-1,\delta+1}$ be given by $\xi \longmapsto \mathcal{L}_\xi g$. If $p>n/(n-2)$, $s>2+n/p$, $0 \leq \delta < n-2-n/p$ and $g \in \mathcal{R}^p_{s,\delta}$ then $\mathrm{div}_g \circ K_g : X^p_{s,\delta} \longrightarrow X^p_{s-2,\delta+2}$ is an isomorphism. ($(\mathrm{div}_g c)_i = -\nabla^j c_{ij}$ for a symmetric 2-tensor c.)

Proof: (Cantor (1979), Choquet-Bruhat et al. (1977a)) The proof is similar in spirit to the above. Since $(\mathrm{div}_e \circ K_e \xi)_j = \partial_i(\partial_i \xi_j + \partial_j \xi_i) = \chi_j \in X^p_{s-2,\delta+2}$ has a solution given by $\xi_j = \Delta_e^{-1}(\chi_j - \frac{1}{2}\partial_i \partial_j \Delta_e^{-1} \chi_i) \in X^p_{s-2,\delta+2}$, $\mathrm{div}_e \circ K_e$ is an isomorphism. It is easily verified that $\mathrm{div}_g \circ K_g$ satisfies the hypothesis of theorem 3.4. Using a metric $g_R = f_R g + (1-f_R)e$ where f_R is a C^∞ function which equals one on a ball of radius R and vanishes outside a ball of radius R+1 Cantor shows that $A_{g_R} = \mathrm{div}_{g_R} \circ K_{g_R} \longrightarrow \mathrm{div}_g \circ K_g$ in the operator norm and, using some technical lemmas, that A_{g_R} is an injection. (See Choquet-Bruhat et al. (1979) for some other methods of proof.)

■

The third usage of theorem 3.7 is to obtain splitting theorems, or more specifically, to obtain a splitting of the tangent space of X at σ. This splitting turns out to give one subspace as the tangent to the orbit through σ so the other orthogonal subspace can be taken to be the tangent of a slice submanifold. The two splitting theorems we require follow directly from corollaries 3.8 and 3.9 and the following lemma.

Lemma 3.10: Let E, F and G be Banach spaces and $f:E\longrightarrow F$, $h:F\longrightarrow G$ bounded linear maps. Then if $h\circ f$ is an isomorphism, we have $F=fE\oplus\ker(h)$.

Corollary 3.11: If p, δ, s and g are as in corollary 3.8 then

$$X^p_{s-1,\delta+1}=d(M^p_{s,\delta})\oplus\mathcal{J}_g$$

where $\mathcal{J}_g=\{\xi\in X^p_{s-1,\delta+1}|\operatorname{div}_g\xi=0\}$.

Corollary 3.12: If p, δ, and g are as above with $s>2+n/p$ then

$$S^p_{s-1,\delta+1}=K_g(X^p_{s,\delta})\oplus J_g$$

where $J_g=\{c\in S^p_{s-1,\delta+1}|\operatorname{div}_g c=0\}$.

The first corollary was first given by Cantor (1979). The second corollary is often called the Berger-Ebin decomposition as it is one of the decompositions for symmetric covariant $W^{s,p}$ 2-tensors on a compact manifold that they obtained using some elliptic operator theory (Berger and Ebin (1969)). (Note that div_g is the formal adjoint of K_g). This decomposition was used by Ebin (1970) in proving the slice theorem for the $W^{s,p}$ Riemannian metrics on a compact manifold in an analogous manner in which corollary 3.12 is used below.

We are now ready to investigate the orbits of $\mathcal{D}^p_{s,\delta}$ and the slices (when they exist) of $\mathcal{R}^p_{s-1,\delta+1}$.

Theorem 3.13: Let $p>n/(n-2)$, $s>2+n/p$ and $0\le\delta<n-2-n/p$. Let $\mathcal{D}^p_{s,\delta}$ act on $\mathcal{R}^p_{s-1,\delta+1}$. Then

(i) For any $g\in\mathcal{R}^p_{s-1,\delta+1}$ the only isometry of g in $\mathcal{D}^p_{s,\delta}$ is the identity I.

(ii) If $g\in R^p_{s-1+k,\delta+1}$, the orbit through g, $\mathcal{O}_g=\{\varphi^*(g)|\varphi\in\mathcal{D}^p_{s,\delta}\}$ is a C^k submanifold.

(iii) If $g\in R^p_{s,\delta+1}$, there is a neighborhood V of I in $\mathcal{D}^p_{s,\delta}$ and a slice of the action, i.e. a submanifold $\mathcal{S}$ of $\mathcal{R}^p_{s-1,\delta+1}$ containing g such that $(\varphi,g)\longmapsto A(\varphi,g)$ is a homeomorphism of $V\times\mathcal{S}$ onto a neighborhood W of g in $\mathcal{R}^p_{s-1,\delta+1}$ and $\mathcal{O}_g\cap\mathcal{S}=\{g\}$.

Proof: (Cantor (1979)) (i) Since $\ker K_g\subset\ker(\mathrm{div}_g\circ K_g)=\{0\}$, the set of infinitesimal isometries is trivial so the isometries of g are isolated. Cantor then shows that the isometry group of g has no nontrivial compact subgroup. The idea behind this argument is that if there were such a subgroup it would have finite order so there would be an isometry $\varphi\in\mathcal{D}^p_{s,\delta}$ ($\varphi\neq I$) such that $\varphi^k=I$) for some k. The asymptotic properties are then used to show that there is an entire neighborhood of fixed points of φ. But any C^1 isometry which fixes a neighborhood is the identity so the subgroup is trivial. Thus all of the orbits of any isometry $\varphi \in \mathcal{D}^p_{s,\delta}$ must be unbounded since otherwise φ would generate a compact subgroup (Kobayashi and Nomizu (1963) p.47,48). Along an orbit, however, the displacement function of an isometry remains constant and thus for φ the displacement must vanish and $\varphi = I$.

This result for the infinitesimal isometries generated by a Killing vector field was obtained independently by Choquet and Choquet-Bruhat (1978) for the asymptotically flat, $n=3$ case, requiring only that the Killing vector field goes to zero at infinity. We will have occasion to use this result as well as corollary 3.9 in showing that certain Killing vector fields vanish as this result does not require as much differentiability.

(ii) Since there are no non-trivial isometries in $\mathcal{D}^p_{s,\delta}$ the map $A_g : \mathcal{D}^p_{s,\delta} \longrightarrow \mathcal{R}^p_{s-1,\delta+1}$ given by $A_g(\varphi) = \varphi^*(g)$ is injective and for $g \in R^p_{s-1+k,\delta+1}$ it is C^k. For $\varphi_0 \in \mathcal{D}^p_{s,\delta}$ and φ near φ_0, $\varphi \circ \varphi_0^{-1}$ is near I and $\varphi^*(g) = (\varphi \circ \varphi_0^{-1})^* \circ \varphi_0^*(g)$ so it is sufficient to show that for $k \geq 1$ $A_g'(I)$ is injective and its image splits (theorem 3.3). Let $\xi \in T_I \mathcal{D}^p_{s,\delta} = X^p_{s,\delta}$ and let φ_t be the flow of ξ. It is fairly straightforward to show that then $\varphi_t \in \mathcal{D}^p_{s,\delta}$ (Cantor (1979) appendix). Then $A_g'(I)\xi = (d/dt)(\varphi_t^*(g)) = K_g \xi = \mathcal{L}_\xi g$. From the Berger-Ebin decomposition (corollary 3.12) and $\ker K_g = \{0\}$ it follows that A_g is a C^k-imbedding so $\mathcal{O}_g$ is a C^k submanifold.

(iii) It is clear that there is an open neighborhood Z of 0 in J_g such that $g+c \in \mathcal{R}^p_{s-1,\delta+1}$ for $c \in Z$. Then

$$\mathcal{S}_g := \{g+c \mid c \in Z\} \tag{3.12}$$

is an imbedded submanifold of $\mathcal{R}^p_{s-1,\delta+1}$. Defining

$\hat{A} : \mathcal{D}^p_{s,\delta} \times \mathcal{S}_g \longrightarrow \mathcal{R}^p_{s-1,\delta+1} : (\varphi, g+c) \longmapsto \varphi^*(g+c)$ we see its tangent at (I,g) is given by

$\hat{A}'(I,g) : X^p_{s,\delta} \oplus J_g \longrightarrow S^p_{s-1,\delta+1} : (\xi, c) \longmapsto K_g \xi + c$ which in view of theorem 3.6 and corollary 3.12 is continuous, onto and injective. The inverse function theorem and its corollaries (c.f. Lang (1972)) give the required homeomorphism. ∎

In order to investigate curves of physically distinct solutions of Einstein's equations in the slice we need to investigate the solutions to the linearized equations off of the spherical background. This is clearly impossible to do directly by solving the equations on all possible backgrounds. Fortunately, however, Nirenberg and Walker (1973) have shown that for elliptic operators which are "differentiably close" to elliptic operators with constant coefficients the dimension of the kernel is locally non-increasing. In other words a sufficiently small perturbation of such an operator gives an operator whose kernel is never larger than the kernel of the unperturbed operator. Thus the dimension of the solution space $\ker \mathcal{L}'(\sigma)$ will not increase for σ close enough to σ. This is used together with Cantor's isomorphism theorem and the fact that ker $\mathcal{L}'(\sigma)$ is 0–dimensional (1–dimensional) in the static (stationary) case, which we prove in chapter 4 (5), to obtain the desired uniqueness theorems.

The relevant theorem, stated below, is the second of the two main theorems about such elliptic operators which we require. It is essentially an adaptation of theorem 4.2 of Nirenberg and Walker (1973) to the weighted Sobolev spaces so we will refer to it as Nirenberg and Walker's theorem. The proof, given in detail for the first time below, follows closely the method of proof given by Nirenberg and Walker which was also followed by Cantor in obtaining the closed range result of theorem 3.7.

Theorem 3.14: If n, k, p, s, δ, A_∞ and A are as in theorem 3.7 then there exists an $\varepsilon>0$ such that if

$$\overline{A}=A_\infty+\sum_{|\mu|\le k}\overline{b}_\mu(x)D^\mu$$

is another elliptic operator for which

$$\|b_\mu-\overline{b}_\mu\|_{p,s-k,k-|\mu|}<\varepsilon \qquad (|\mu|\le k)$$

then the dimension of $\ker\overline{A}$ is less than or equal to the dimension of $\ker A$.

Proof: Denote the dimension of $\ker A$ by q and suppose the theorem is false. Then for all integers $i>0$ there exists an elliptic operator $A_i{:}M^p_{s,\delta}\longrightarrow M^p_{s-k,\delta+k}$ given by $A_i=A_\infty+\sum_{|\mu|\le k}b^i_\mu(x)D^\mu$.

such that $\|b_\mu - b_\mu^i\|_{p,s-k,k-|\mu|} < i^{-1}$ for $|\mu| \le k$ and such that dim ker $A_i > q$. A positive number ε_0 can be found such that any subspace of $M^p_{s,\delta}$ with dimension greater than q contains an element of norm 1 whose distance from ker A is at least ε_0. Choose such a f_i in each ker A_i, $\|f_i\|_{p,s,\delta}=1$. Equation (3.11) of theorem 3.7 implies $\|Af_i\|_{p,s-k,\delta+k} = \|(A-A_i)f_i\|_{p,s-k,\delta+k} < C_1 i^{-1} \|f_i\|_{p,s,\delta}$ for some C_1 so $\|Af_i\|_{p,s-k,\delta+k} \longrightarrow 0$. Therefore, if a subsequence of $\{f_i\}$ exists which is Cauchy in $M^p_{s,\delta}$, since A is closed the subsequence converges to an element in ker A. But all the $\{f_i\}$ lie a distance ε_0 away from ker A. Thus to prove the theorem we need only find a Cauchy subsequence of $\{f_i\}$.

Let $\varphi_R : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a C^∞ function with compact support such that $\varphi_R(x)=1$ if $|x| \le R$, $\varphi_R(x)=0$ if $|x| \ge 2R$ and $|D^\mu \varphi_R(x)| \le 1$ for all μ. Write $f_i = \varphi_R f_i + (1-\varphi_R) f_i$ and note that $\{f_i\}$ is Cauchy if $\{\varphi_R f_i\}$ and $\{(1-\varphi_R) f_i\}$ are Cauchy for some R. Let $B_R = \{x \in \mathbb{R}^n \mid |x| \le R\}$.

For all R, the sequence $\{\varphi_R f_i\}$ is bounded in $W^{s,p}(B_{2R})$. This together with the Rellich compactness theorem stated below implies it has a subsequence (taken to be all of $\{\varphi_R f_i\}$) which converges to an element in $L^p(B_{2R})$.

Lemma 3.15: (Rellich compactness lemma) Let $0 \le r < s$. Then any bounded sequence in $W^{s,p}(B_{2R})$ has a convergent subsequence in $W^{r,p}(B_{2R})$.

Proof: (c.f. Friedman (1969) p.31)

Now $\|A(\varphi_R f_i)\|_{p,s-k} \le \|\varphi_R A f_i\|_{p,s-k} + \|A(\varphi 1 R f_i) - \varphi_R A f_i\|_{p,s-k}$ and $\{A(\varphi_R f_i) - \varphi_R A f_i\}$ has support in B_{2R} and is bounded in $W^{s-k+1,p}(B_{2R})$ since the highest derivative term of f_i cancels. The Rellich compactness lemma together with the fact that $\{\varphi_R A f_i\}$ is clearly Cauchy implies $\{A(\varphi_R f_i)\}$ is Cauchy in $L^p(B_{2R})$.

As long as $\lim_{|x|\to\infty} \sup |\sigma^{k-|\mu|} b_\mu(x)|$ for $|\mu| \le k$ is small enough or $b_\mu(x)$ is uniformly continuous on $\mathbb{R}^n$ for $|\mu| = k$ there is a constant C_2 such that $\|f\|_{p,s} \le C_2(\|Af\|_p + \|f\|_p)$ (c.f. Nirenberg and Walker (1973), equation (4.1)). Since the former condition is always satisfied (c.f. proof of theorem 3.7) (and the latter is if $s > n/p+1$ since then b_μ is C^1 for $|\mu| = k$) we have

$$\|\varphi_R f_i - \varphi_R f_j\|_{p,s} \le C_2(\|A(\varphi_R f_i) - A(\varphi_R f_j)\|_p + \|\varphi_R f_i - \varphi_R f_j\|_p)$$

so $\{\varphi_R f_i\}$ is Cauchy in $W^{s,p}(B_{2R})$ and it follows that $\{\varphi_R f_i\}$ is Cauchy in $M^p_{s,\delta}$ for each R.

Since $A_\infty : M^p_{s,\delta} \longrightarrow M^p_{s-k,\delta+k}$ is an isomorphism (lemma 1.1 of Cantor (1979)) $\|f\|_{p,s,\delta} \le C_3 \|A_\infty f\|_{p,s-k,\delta+k}$ for all $f \in M^p_{s,\delta}$. Thus, for all $R \ge 1$,

$$\|(1-\varphi_R) f_i\|_{p,s,\delta} \le C_3(\|A(1-\varphi_R) f_i\|_{p,s-k,\delta+k} + \|(A_\infty - A)(1-\varphi_R) f_i\|_{p,s-k,\delta+k}).$$

As seen in the proof of theorem 3.7, $\lim_{R\longrightarrow\infty} \|(A_\infty - A)(1-\varphi_R) f_i\|_{p,s-k,\delta+k}$

$\longrightarrow 0$ so choosing R large enough we have

$$\|(1-\varphi_R)f_i\|_{p,s,\delta} \le C_4 \|A(1-\varphi_R)f_i\|_{p,s-k,\delta+k}$$

$$\le C_4(\|(1-\varphi_R)Af_i\|_{p,s-k,\delta+k} + \|A(1-\varphi_R)f_i-(1-\varphi_R)Af_i\|_{p,s-k,\delta+k}).$$

The sequence $\{A(1-\varphi_R)f_i-(1-\varphi_R)Af_i\}$ has support in B_{2R} and is bounded in $W^{s-k+1,p}(B_{2R})$, so is Cauchy in $W^{s-k,p}(B_{2R})$. By passing to a subsequence we can assume it is Cauchy in $M^p_{s-k,\delta+k}$. Since $\{(1-\varphi_R)Af_i\}$ is clearly Cauchy in $M^p_{s-k,\delta+k}$. $\{(1-\varphi_R)f_i\}$ is Cauchy in $M^p_{s,\delta}$. ∎

CHAPTER IV

ON THE UNIQUENESS PROBLEM FOR STATIC PERFECT FLUIDS

4.1 Introduction

The results in this chapter for the case of a discontinuous matter density at the star boundary first appeared in Künzle and Savage (1980b). The extension to more general equations of state is new. As indicated in §2.2 we conjecture that the set $S_{\rho(p)}$ of static bounded perfect fluid solutions to Einstein's equations with a fixed equation of state $\rho(p)$ satisfying conditions (i) to (v) of §2.4 (and fixed surface temperature T_b) with U having only one (nondegenerate) critical point consists of the spherical solutions. Recall that we are going to consider the set S_ρ as the inverse image of a differentiable map $\mathscr{L}:X\longrightarrow Y$ between some suitable weighted Sobolev spaces. Since all static solutions can be described by a positive definite metric γ and a gravitational potential function U on Σ, which by the argument about spherical solutions can be taken to be diffeomorphic to $\mathbb{R}^3$, we want to consider S_ρ as a subset of an appropriate Banach space formed from the set $\{(\gamma,U)\}$.

In view of the asymptotic conditions (1.35) we would expect U to be in $M^p_{s,\delta}$ and γ to be in $\mathcal{R}^p_{s,\delta}$ for $0\leq\delta<1-3/p$ and some s and p. But the linearization of $\mathscr{L}$ on a background $\sigma\in S_\rho$

i.e. $\mathscr{L}'(\sigma)$, does not approach a system with constant coefficients as σ tends to the trivial flat space solution. In other words, for a fixed $\rho(p)$ a curve $m:[0,m_{crit}]\longrightarrow\sigma=(\gamma,U)\in S_\rho$ which tends to the flat solution $(\gamma_{ij}=\delta_{ij},U=0)$ as $m\longrightarrow 0$ is continuous but is not differentiable at $m=0$. Thus Cantor's isomorphism theorem cannot be applied, as one would expect since $\mathscr{L}'(\sigma)$ is physically expected to have a 1-dimensional kernel on this larger space corresponding to a curve of solutions parameterized by the mass. For this and other technical reasons which will become apparent when we deal with the appropriate slicing theorem it is more convenient to fix the total mass and consider only the subset $S_{\rho,m}=\{\sigma\in S_\rho|\mathrm{mass}(\sigma)=m\}$. Then for any two σ_1, $\sigma_2\in S_{\rho,m}$ the difference U_1-U_2 has a fall off rate faster than $1/|x|$.

We expect $U_b=U(p=0)$ to be determined by m and $\rho(p)$ but as noted in §2.3 we cannot yet be sure. Therefore we will describe the set $S_{\rho,m}$ by the triples $\sigma=(\gamma,U,U_b)$. There is clearly a unique spherically symmetric solution in this set, which we will denote by σ. By choosing the appropriate Banach space structure for $S_{\rho,m}$ we will be able to investigate the behaviour of curves of physically distinct solutions to Einstein's equations with constant mass m, fixed equation of state $\rho(p)$ and fixed surface temperature T_b.

4.2 Choosing a Banach manifold structure on the set $\{(\gamma, U, U_b)\}$

We are going to want to apply a slicing theorem based on theorem 3.13 to the space which we pick. But for $0 \leq \delta < 1-3/p$ theorem 3.13 is applicable only for $g \in R^p_{s,1+\delta}$ (for some s and p) whereas the asymptotic conditions (1.33) imply $\gamma \in \mathcal{R}^p_{s,\delta}$. This is due to the fact that on $\mathbb{R}^3$ decomposing tensors which fall off as $1/r$ is usually impossible; $1/r^2$ fall off is generally required. Fortunately there is enough coordinate freedom to avoid this problem.

In cartesian coordinates the asymptotic conditions for the general stationary metric give

$$\gamma_{ij}=\delta_{ij}+\gamma^1_{ij}|x|^{-1}+O(|x|^{-2}), \quad h_i=h^1_i|x|^{-2}+O(|x|^{-3}), \quad U=-m|x|^{-1}+O(|x|^{-2})$$

with $x^k\partial_k\gamma^1_{ij}=0$, $x^k\partial_k h^1_i=0$. But it is well known that for stationary spacetimes the quantities γ^1_{ij} and h^1_i can be made to vanish by a suitable coordinate transformation that is asymptotically Euclidean (i.e. consisting of a rotation and a translation). For example γ^1_{ij} and h^1_i can be shown to vanish by demanding that the coordinates x^i be harmonic so $\gamma^{ij}\Gamma^k_{ij}=0$ (Misner, Thorne and Wheeler (1972) p.456) or, in the static case, by demanding that the coordinates are asymptotically harmonic, $\gamma^{ij}\Gamma^k_{ij}=O(|x|^{-3})$ (Beig (1979)). (This latter result can probably be extended to the stationary case.) Choquet-Bruhat et al. (1979) have dealt with this problem by restricting to the set $\{g \in \mathcal{R}^{p}_{s,\delta} \,|\, \gamma^{ij}\Gamma^{k}_{ij}=0\}$ which near flat spacetime can be shown to be a submanifold. Whether this holds true far from flat space is not obvious and we will not take this route but as we will see at the end of chapter 5 this may be one way of attempting to obtain a surjective map. It is not necessary to make a restriction to harmonic coordinates since the desired falloff can be obtained explicitly using the (2+1)-dimensional formalism where U is treated as an intrinsically defined coordinate. This argument depends essentially on the surfaces of constant U being topologically 2-spheres and involves making a coordinate transformation on these 2-spheres. From now on we will therefore consider the coordinate system of $\Sigma \cong \mathbb{R}^3$ chosen in such a way that

$$\gamma_{ij}=\delta_{ij}+O(|x|^{-2}), \quad h_i=O(|x|^{-3}), \quad U=-m/|x|+O(|x|^{-2}) \tag{4.1}$$

Further coordinate transformations that are asymptotic to the identity will then not destroy this behaviour.

Another problem arises due to the differentiability at the star boundary. Recall that for an equation of state such that $\lim_{p \to 0} \rho p^{-\varepsilon}=d>0$ for $(\ell-1)/\ell<\varepsilon\leq\ell/(\ell+1)$ that $M \in C^k$ for $k<\ell$ but $\lim_{p \to 0} |d^{\ell}M/dU^{\ell}|=\infty$ unless $\varepsilon=\ell/(\ell+1)$ in which case M is $C^{\ell-1}$ piecewise C^{ℓ} (theorem 2.1). Using the derivatives calculated in

§2.4 it is easy to verify that

$$\|M\|_{p,\ell,\delta+n} \le C_1 + C_2 \int_0^{p_c} \overline{p}^{((\ell+1)\varepsilon-\ell)p} (\rho+\overline{p})^{-1} d\overline{p} \qquad (4.2)$$

for some positive constants C_1 and C_2 and any $n>0$ since the only unbounded integral is the ℓ^{th} normal derivative to the star boundary. The integral on the left side of (4.2) is bounded if $p<(1-\varepsilon)/(\ell-(\ell+1)\varepsilon)$. In order to apply the theorems of the previous chapter we must take $p>3$. Thus for ε such that $(3\ell-1)/(3\ell+2)<\varepsilon\le\ell/(\ell+1)$ $M\in M^p_{\ell,\delta+n}$ (and $M\notin M^p_{\ell+1,\delta+n}$) for $3<p<(1-\varepsilon)/(\ell-(\ell+1)\varepsilon)$ and for all $p>3$ if $\varepsilon=\ell/(\ell+1)$). If $(\ell-1)/\ell<\varepsilon\le(3\ell-1)/(3\ell+2)$ and $p>3$ then $M\in M^p_{\ell-1,\delta+n}$ (and $M\notin M^p_{\ell,\delta+n}$). We can now choose the equation of state, and thus ε, to be fixed once and for all so that $M\in M^p_{s-3,\delta+n}$, $M\notin M^p_{s-2,\delta+n}$ for some fixed $s\ge3$, for any $p>3$ satisfying the above conditions for an upper bound which depends on the value of ε, and for any large n.

This is partly why we have chosen $\rho(p)$ to be C^ℓ piecewise $C^{\ell+1}$ in the interior region. Thus even though p is uniquely determined by $dp+(\rho+p)dU=0$ when U_b is given, if $\rho(p)$ is piecewise C^1 on $(0,p_c]$ our formalism does not allow the incorporation of different differentiabilities in different regions of Σ. The other reason is that in order to obtain a slice at the spherically symmetric solution we have to define the topology in terms of a smoothed out potential which is a little more differentiable at the spherical solution. Allowing less differentiability of $\rho(p)$ would mean that a different smoothing potential would have to be constructed for different equations of state. This is therefore not a fundamental barrier but would be messy to deal with. However, as we noted before, it is not a great physical restriction to assume as much differentiability as we want for the equation of state.

Now, using the asymptotic conditions (4.1) and taking $0 \leq \delta < 1-3/p$, Einstein's equations (1.29, 1.30) and the weighted Sobolev space properties (3.6–3.8) indicate that $U \in M^p_{s-1,\delta}$ but that we could choose $\gamma \in R^p_{s,\delta+1}$. If we took the set $\mathcal{R}^p_{s-1,\delta+1} \oplus M^p_{s-1,\delta}$ to model the space $\{(\gamma,U)\}$ on, the action of the diffeomorphism group $\mathcal{D}^p_{s,\delta}$ on this would then only be C^0, as we will see, even though for $\gamma \in R^p_{s,\delta+1}$, the action of $\mathcal{D}^p_{s,\delta}$ on $\mathcal{R}^p_{s-1,\delta+1}$ is C^1. A C^0 action is not enough to obtain a slice theorem so we must describe the topology of our set $\{(\gamma,U)\}$ in terms of a modified potential $\hat{U}$ which we make to be in $M^p_{s,\delta}$ for the spherical solution.

The potential U is not in $M^p_{s,\delta}$ because the normal derivative of order $s-2$ of M at the star boundary is not in $M^p_{0,\delta+n}$, as can be seen from equation (1.30). Define

$$\hat{U}=U-f\tilde{p}^{2-\epsilon} \tag{4.3}$$

for some function f on Σ. Let $[\nabla_{\nabla U}\cdots\nabla_{\nabla U}\hat{U}]_s=\lim_{U\to U_b^-}\nabla_{\nabla U}\cdots\nabla_{\nabla U}\hat{U}$ $-\lim_{U\to U_b^+}\nabla_{\nabla U}\cdots\nabla_{\nabla U}\hat{U}$. As seen in §2.4 this is proportional to the difference in the s^{th} normal derivative of $\overset{\frown}{U}$ with respect to a C^{∞} coordinate at the star boundary. Doing similar calculations to those of §2.4, using $d\tilde{p}/dU=-2M$, one finds that

$$[\nabla_{\nabla U}\cdots\nabla_{\nabla U}\overset{\frown}{U}]_s=\lim_{p\to 0}(2W^{-2}+fde^{-2U(1-\varepsilon)}(2-\varepsilon))d^{s-2}M/dU^{s-2} \qquad (4.4)$$

so that $\overset{\frown}{U}\in M^p_{s,\delta}$ provided $\lim_{U\to U_b^-}=(2(2-\varepsilon)dW^2(U_b)d^{-2U_b(1-\varepsilon)})^{-1}$ and $f\geq 0$ for $U\leq U_b$. Since it is not easy to choose such an f that is a simple functional of U and γ and regular in the whole interior domain (recall $W(U_c)=0$) we let $f=f_0=(2(2-\varepsilon)dW^2(U_b)e^{-2U_b(1-\varepsilon)})^{-1}=$ const. so that for the spherically symmetric solution $\hat{U}\in M^p_{s,\delta}$, $\gamma\in R^p_{s,1+\delta}$. This is also convenient for the stationary case so in the following we will always consider f_0 to be a fixed constant determined by the unique spherically symmetric $\sigma\in S_{\rho,m}$.

For the remainder of this chapter we take $0\leq\delta<1-3/p$ determined by the asymptotic conditions, $s\geq 3>2+n/p$ fixed by the equation of state and $p>3$ with an upper bound determined by the equation of state as seen previously. Define

$$\mathcal{P}^p_{s-1,\delta+1}:=\{\sigma=(\gamma_{ij},U,U_b)|\gamma_{ij}\in\mathcal{R}^p_{s-1,\delta+1},\overset{\frown}{U}-\hat{U}\in M^p_{s-1,\delta+1},U_b\in\mathbb{R}\} \qquad (4.5)$$

where $\overset{\frown}{U}$ and $\hat{U}$ are defined by (4.3) in terms of U and U, respectively. Since U and $\overset{\frown}{U}$ determine each other for given U_b we will indiscriminately write $\sigma=(\gamma,\overset{\frown}{U},U_b)$ and $\sigma=(\gamma,U,U_b)$. We can calculate with either $\overset{\frown}{U}$ or U but the topology is defined in terms of the former. $\mathcal{P}^p_{s-1,\delta+1}$ is clearly a Banach manifold with any asymptotically flat solutions of Einstein's equations in $\mathcal{P}^p_{s-1,\delta+1}$ having the same mass since for any σ_1, $\sigma_2\in\mathcal{P}^p_{s-1,\delta+1}\cap S_\rho$ we have found $\hat{U}_1-\hat{U}_2$ ($=U_1-U_2$ in vacuo) to fall off at a rate greater than $1/r$. In this sense the fixed mass just becomes a boundary condition which we incorporate into the space $\mathcal{P}^p_{s-1,\delta+1}$, but note that $\mathcal{P}^p_{s-1,\delta+1}$ is much larger than the space of asymptotically flat solutions.

We are now ready to investigate the action of the diffeomorphism group $\mathcal{D}^p_{s,\delta}$ on $\mathcal{P}^p_{s-1,\delta+1}$ and to see under what conditions there is a slice of the action. The following theorem is a direct analogue for the space $\mathcal{P}^p_{s-1,\delta+1}$ of theorems 3.6 and 3.13.

Theorem 4.1: (i) $\mathcal{D}^p_{s,\delta}$ acts continuously on $\mathcal{P}^p_{s-1,\delta+1}$ by

$$A:\mathcal{D}^p_{s,\delta}\times\mathcal{P}^p_{s-1,\delta+1}\longrightarrow\mathcal{P}^p_{s-1,\delta+1}:(\varphi,(\gamma,\hat{U},U_b))\longmapsto(\varphi^*(\gamma),\hat{U}\circ\varphi,U_b).$$

Moreover, $A_\varphi:\sigma\longmapsto A(\varphi,\sigma)$ is C^∞ and, if $\sigma\in\mathcal{P}^p_{s-1+k,\delta+1}$, $A_\sigma:\varphi\longmapsto A(\varphi,\sigma)$ is C^k for $k=0$ or 1.

(ii) If $\sigma\in\mathcal{P}^p_{s-1+k,\delta+1}$ (for $k=0$ or 1) then the orbit $\mathcal{O}_\sigma:=\{A(\varphi,\sigma)|\varphi\in\mathcal{D}^p_{s,\delta}\}\subset\mathcal{P}^p_{s-1,\delta+1}$ is a C^k-submanifold.

(iii) If $\sigma\in\mathcal{P}^p_{s,\delta+1}$ there is a neighborhood V of I in $\mathcal{D}^p_{s,\delta}$ and a slice of the action, i.e. a submanifold $\mathcal{S}$ of $\mathcal{P}^p_{s-1,\delta+1}$ containing σ such that $(\varphi,\sigma')\longrightarrow A(\varphi,\sigma')$ is a homeomorphism of $V\times\mathcal{S}$ onto a neighborhood W of σ in $\mathcal{P}^{p}_{s-1,\delta+1}$ and $\mathcal{O}_{\sigma}\cap\mathcal{S}=\{\sigma\}$.

Proof: (i) Let $\sigma=(\gamma,\hat{U},U_b)\in\mathcal{P}^{p}_{s-1,\delta+1}$. Since $\gamma\in\mathcal{R}^{p}_{s-1,\delta+1}$ theorem 3.6 gives (i) as far as the metric is concerned. The function $\hat{U}$ is explicitly known in the vacuum region (from (2.4)) and it is clear that $\sigma(x)^{(|\mu|+\delta-1)}D^{\mu}\hat{U}$ is bounded for $0\leq|\mu|\leq s$. Thus theorem 3.5 shows that $\mathcal{D}^{p}_{s,\delta}\oplus M^{p}_{s-1,\delta+1}(\hat{U})\longrightarrow M^{p}_{s-1,\delta+1}(\hat{U}):(\varphi,\hat{U})\longmapsto\hat{U}\circ\varphi$ is continuous (and linear whence C^{∞} in $\hat{U}$). But if $\sigma\in\mathcal{P}^{p}_{s,\delta+1}$ then $\hat{U}\in M^{p}_{s,\delta+1}(\hat{U})\subset M^{p}_{s,\delta}$. Therefore $\partial_k\hat{U}\in M^{p}_{s-1,\delta+1}$ and $(\varphi,\partial_k\hat{U})\longmapsto(\partial_k\hat{U})\circ\varphi$ is continuous. Hence A_{σ} is C^{1} in φ. Since the action on U_b is trivial this establishes (i).

(ii) Since there are no nontrivial isometries in $\mathcal{D}^{p}_{s,\delta}$ the map A_{σ} is injective. If $k=1$ the tangent map TA_{σ} at $\varphi=I$ is given by $\xi\longmapsto(\mathcal{L}_{\xi}\gamma,\mathcal{L}_{\xi}\hat{U},0)$ where ξ is an $M^{p}_{s,\delta}$ vector field on Σ and $(\mathcal{L}_{\xi}\gamma,\mathcal{L}_{\xi}\hat{U},0)\in S^{p}_{s-1,\delta+1}\oplus M^{p}_{s-1,\delta+1}\oplus\mathbb{R}\cong T_{\sigma}\mathcal{P}^{p}_{s-1,\delta+1}$. But Cantor's version of the Berger–Ebin decomposition, corollary 3.12, for $\gamma\in R^{p}_{s,\delta+1}$, gives the splitting $S^{p}_{s-1,\delta+1}=K_{\gamma}(X^{p}_{s,\delta})\oplus J_{\gamma}$ where $J_{\gamma}=\{c\in S^{p}_{s-1,\delta+1}\,|\,\mathrm{div}_{\gamma}c=0\}$.

Thus, for $\sigma\in\mathcal{P}^{p}_{s,\delta+1}$, given $(c,u)\in S^{p}_{s-1,\delta+1}\oplus M^{p}_{s-1,\delta+1}$ there are unique $\xi\in X^{p}_{s,\delta}$ and $l\in S^{p}_{s-1,\delta+1}$, with $\mathrm{div}_{\gamma}l=0$ such that $c=K_{\gamma}(\xi)+l$. If we let $v=u-\mathcal{L}_{\xi}\hat{U}$ then $(c,u)=(\mathcal{L}_{\xi}\gamma,\mathcal{L}_{\xi}\hat{U})+(l,v)$ which gives the direct sum decomposition of $S^{p}_{s-1,\delta+1}\oplus M^{p}_{s-1,\delta+1}$ (since $(c,u)=0$, $\mathrm{div}_{\gamma}l=0$ and $l=\mathcal{L}_{\xi}\gamma\Longrightarrow\mathrm{div}_{\gamma}\circ K_{\gamma}(\xi)=0$ whence $\xi=0$ whence $l=0$ and $v=0$). Thus the image of T_1A_σ splits and A_σ is a C^1-imbedding.

(iii) Choose an open neighborhood B of 0 in J_γ such that $\gamma+l$ is positive definite for $l\in B$. Now

$$\mathcal{S}:=\{(\gamma+l,\hat{U}+u,U_b+k)|l\in B,\ u\in M^p_{s-1,\delta+1},\ k\in\mathbb{R}\}$$

is clearly an imbedded submanifold of $\mathcal{P}^p_{s-1,\delta+1}$. Defining $\hat{A}:\mathcal{D}^p_{s,\delta}\times\mathcal{S}\longrightarrow\mathcal{P}^p_{s-1,\delta+1}:(\varphi,(\gamma+l,\hat{U}+u,U_b+k))\longmapsto(\varphi^*(\gamma+l),\varphi^*(\hat{U}+u),U_b+k)$ we have for its tangent at (I,σ), $T_{(I,\sigma)}\hat{A}:X^p_{s,\delta}\oplus J_\gamma\oplus M^p_{s-1,\delta+1}\oplus\mathbb{R}\longrightarrow S^p_{s-1,\delta+1}\oplus M^p_{s-1,\delta+1}\oplus\mathbb{R}:(\xi,L,V,\kappa)\longmapsto(K_\gamma\xi+L,\mathcal{L}_\xi\hat{U}+V,\kappa)$ which is clearly continuous and onto in view of corollary 3.12. It is injective, for $K_\gamma\xi+l=0$ implies $\xi=0$, $l=0$ whence $\mathcal{L}_\xi\hat{U}=0$. The inverse function theorem and its corollaries then imply (iii). ■

In particular, the tangent space to $\mathcal{P}^p_{s-1,\delta+1}$ at $\sigma\in P^p_{s,\delta+1}$ splits, i.e.

$$T_\sigma\mathcal{P}^p_{s-1,\delta+1}=T_\sigma\mathcal{O}_\sigma\oplus T_\sigma\mathcal{S}. \tag{4.6}$$

(This theorem has a trivial extension to $\sigma\in\mathcal{P}^p_{s-1+k,\delta+1}$ for all $k\geq 2$ but such differentiable spacetimes cannot be in $S_{\rho,m}$ for our fixed equation of state so we are not interested in them.)

We would like to know the differentiable structure of the set $S_{\rho,m}\cap\mathcal{P}^p_{s-1,\delta+1}$ of static solutions of Einstein's equations with a fixed mass m, fixed surface temperature T_b and a fixed equation of state. From Einstein's equations (1.29,1.30) it is clear that this set can be characterized by the inverse image of 0 under the map

$$\mathcal{L}:\mathcal{P}^{p}_{s-1,\delta+1}\longrightarrow S^{p}_{s-3,\delta+3}\oplus M^{p}_{s-3,\delta+3}=:\mathcal{Q}$$

$$(\gamma_{ij},U,U_{b})\longmapsto(R_{ij}-2\partial_{i}U\partial_{j}U+2\tilde{\mathfrak{p}}\gamma_{ij},\Delta U-M). \tag{4.7}$$

recalling that $\tilde{\mathfrak{p}}$ and M are uniquely determined functions of U if $\rho(p)$ and U_b are given.

Physical intuition leads to the conjecture that $\mathcal{L}^{-1}(0)=\mathcal{O}_{\sigma}$, i.e. all solutions in $S_{\rho,m}\cap\mathcal{P}^{p}_{s-1,\delta+1}$ are obtained from a given spherically symmetric one by diffeomorphisms. According to the linearization stability technique we should show that the tangent map $\mathcal{L}'(\sigma)=T_{\sigma}\mathcal{L}:T_{\sigma}\mathcal{P}^{p}_{s-1,\delta+1}\longrightarrow T_{\mathcal{L}(\sigma)}\mathcal{Q}$ is surjective and its kernel splits at the spherical solution σ. It would then follow that $\mathcal{L}^{-1}(0)$ is a submanifold with its tangent space at σ tangent to the orbit $\mathcal{O}_{\sigma}$. This result would not be as strong as the above conjecture but would be a local version of it. As we stated before, the existence of a Berger–Ebin type splitting gives us the splitting of the kernel, namely, equation (4.6). However, this map is not surjective, as we will see. Its kernel, the tangent space to the orbit through σ, is in some sense too large to allow it to map onto $\mathcal{Q}$, or, in some sense, $\mathcal{Q}$ is too large. But it is by no means obvious what sort of a submanifold of $\mathcal{Q}$ to pick to try and make $\mathcal{L}'(\sigma)$ surjective.

However all is not lost. The set of equivalence classes of

$\mathcal{P}^p_{s-1,\delta+1}$ in a neighborhood of σ with respect to the action of $\mathcal{D}^p_{s,\delta}$ is in one-to-one correspondence with the slice $\mathcal{S}$ through σ. By solving the linearized equations on the spherical background and finding that $\ker \mathcal{L}'(\sigma)=0$ and then using Cantor's isomorphism theorem 3.7 and Nirenberg and Walker's theorem 3.14 to conclude that the kernel is zero for σ close to σ we shall show that if there was a C^1-curve $\sigma(\lambda)\,(-\varepsilon,\varepsilon)\longrightarrow S_{\rho,m}\cap\mathcal{S}$ such that $\sigma(0)=\sigma$ then its tangent vector would have to vanish everywhere so that in fact, $\sigma(\lambda)=\sigma$ for all λ. Less precisely, there are no non-constant differentiable (C^1) curves of physically distinct solutions of Einstein's equations with fixed m, T_b and $\rho(p)$ which pass through the spherical solution.

4.3 Linearization of $\mathcal{L}$ to find $\ker \mathcal{L}'(\sigma)$

Let $(c_{ij}=\delta\gamma_{ij},\delta\hat{U},k)\in T_\sigma\mathcal{P}^p_{s-1,\delta+1}=S^p_{s-1,\delta+1}\oplus M^p_{s-1,\delta+1}\oplus\mathbb{R}$. For $\sigma\in\mathcal{P}^p_{s,\delta+1}$ and, in particular, for $\sigma=\sigma$ theorem 3.13 shows that one has the unique decompositions

$$c_{ij}=\Phi_{ij}+\mathcal{L}_\xi\gamma_{ij},\quad \nabla^i\Phi_{ij}=0,\quad \delta\hat{U}=\hat{\Psi}+\mathcal{L}_\xi\hat{U} \tag{4.8}$$

where Φ, $\mathcal{L}_\xi\gamma\in S^p_{s-1,\delta+1}$, $\hat{\Psi}$, $\mathcal{L}_\xi\hat{U}\in M^p_{s-1,\delta+1}$ and $k=\delta(U_b)$ (which is, in general, not equal to $(\delta U)_b$). Since the relation between $\hat{U}$ and U is known we can express $\delta\hat{U}$ in terms of δU by

$$\delta\hat{U}=(1-4f_0M\tilde{p})\delta U+2(2-\varepsilon)kf_0\tilde{p}^{1-\varepsilon}(M-\tilde{p})+(\delta f_0)\tilde{p}^{2-\varepsilon}. \tag{4.9}$$

But since the linearized equations are considerably simpler when expressed in terms of $\delta U =: u =: \mathcal{L}_\xi U + \psi \in M^p_{s-2,\delta+1}$, c and k we will perform all calculations using u, c and k. We need only remember that the differentiability properties at $U=U_b$ must be expressed in terms of $\tilde{U}$.

The variations of $\tilde{p}$ and M which have been used already in obtaining (4.9) are determined in terms of δU and k from the integral equation (2.9) obtained from the equation of hydrostatic equilibrium,

$$\delta\tilde{p}=-2M\delta U+2k(M-\tilde{p}), \quad \delta M=(dM/dU)\delta U-k(2M+dM/dU) \qquad (4.10)$$

We now want to linearize $\mathcal{L}$ in the manner of the linearization procedure in §3.1 where we saw that the variations $\delta\sigma$ can be thought of as the tangent to a curve of solutions. In terms of $\delta\gamma_{ij}=c_{ij}$ the varied Christoffel symbols are then

$$\delta\Gamma^i_{jk}=\tfrac{1}{2}\gamma^{ir}(\nabla_k c_{jr}+\nabla_j c_{kr}-\nabla_r c_{jk})$$

so that the variation of the Ricci tensor becomes

$$\delta R_{ij}=\tfrac{1}{2}\gamma^{rs}(2\nabla_r\nabla_{(i}c_{j)s}-\nabla_i\nabla_j c_{rs}-\nabla_r\nabla_s c_{ij}).$$

Similarly, although we do not use the variation of the volume element until we compute the variation of the angular momentum in chapter 5, note that

$$\delta\det(\gamma_{ij})=\det(\gamma_{ij})c^r_r.$$

A lengthy calculation in the same manner then gives that

$$\mathcal{L}'(\sigma)=(\mathcal{L}'_1(\sigma),\mathcal{L}'_2(\sigma)):T_\sigma\mathcal{P}^p_{s-1,\delta+1}=S^p_{s-1,\delta+1}\oplus M^p_{s-1,\delta+1}\oplus\mathbb{R}\longrightarrow S^p_{s-3,\delta+3}\oplus M^p_{s-3,\delta+3}$$

is given by

$$\begin{aligned}\mathcal{L}'_1(\sigma)(c,u,k)_{ij}&=\tfrac{1}{2}(\nabla^r\nabla_r c_{ij}+\nabla_i\nabla_j c^r_r)\\&-\nabla_{(i}\nabla^r c_{j)r}-3R_{r(i}c^r_{j)}+R^{rs}c_{rs}\gamma_{ij}\\&+\tfrac{1}{2}Rc_{ij}+c^r_r(R_{ij}-\tfrac{1}{2}R\gamma_{ij})-2\tilde{p}c_{ij}\\&+4Mu\gamma_{ij}+4U_{(i}\partial_{j)}u-4k(M-\tilde{p})\gamma_{ij}\end{aligned}\qquad(4.11a)$$

$$\left.\begin{aligned}\mathcal{L}'_2(\sigma)(c,u,k)&=\Delta u-(dM/dU)u+\tfrac{1}{2}U^i\partial_i c^r_r\\&-U^{ij}c_{ij}+k((dM/dU)+2M)-U^i\nabla^j c_{ij}\end{aligned}\right\}\qquad(4.11b)$$

where $U_i:=\partial_i U$, $U_{ij}:=\nabla_i\partial_j U$, $U^i:=\gamma^{ir}U_r$ etc.

Our object is to show that $\mathcal{L}'(\sigma)(c,u,k)=0$ for (c,u,k) tangent to the slice $\mathcal{S}$ implies that $(c,u,k)=0$. We should do this first for $\sigma=\sigma$ and then extend to a neighborhood of σ by the use of Nirenberg and Walker's theorem 3.14. But this theorem is of use only for elliptic operators and it is easily seen that the operator $\mathcal{L}'(\sigma)$ is not elliptic. However we can reduce

$$\mathcal{L}'(\sigma)(c,u,k)=0 \qquad (4.12)$$

to an elliptic, inhomogeneous system if we restrict consideration of this system to the slice submanifold $\mathcal{S}$. To this end, define

$$L_1(\sigma)(c,u):=\mathscr{L}'_1(\sigma)(c,u,k)+\tfrac{1}{2}K_\gamma\circ\mathrm{div}_\gamma c+4k(M-\tilde{p})\gamma \qquad (4.13a)$$

and

$$L_2(\sigma)(c,u):=\mathscr{L}'_2(\sigma)(c,u,k)+\gamma(\nabla U,\mathrm{div}_\gamma c)-k((dM/dU)+2\dot{M}). \qquad (4.13b)$$

It is straightforward to verify that $L(\sigma)=(L_1(\sigma),L_2(\sigma))$: $S^p_{s-1,\delta+1}\oplus M^p_{s-1,\delta+1}\longrightarrow S^p_{s-3,\delta+3}\oplus M^p_{s-3,\delta+3}$ is a second order elliptic operator and that the equation (4.11) is now equivalent to

$$L_1(\sigma)(c,u)-\tfrac{1}{2}K_\gamma\circ\mathrm{div}_\gamma c=kq_1(\sigma) \qquad (4.14a)$$

$$L_2(\sigma)(c,u)-\gamma(\nabla U,\mathrm{div}_\gamma c)=kq_2(\sigma) \qquad (4.14a)$$

where

$$q(\sigma):=(q_1(\sigma),q_2(\sigma)):=(4(M-\tilde{p})\gamma,-(dM/dU)-2M). \qquad (4.15)$$

We will show, for $\sigma\in\mathscr{S}$, that the inhomogeneous equation

$$L(\sigma)(c,u)=kq(\sigma) \qquad (4.16)$$

for $(c,u)\in S^p_{s-1,\delta+1}\oplus M^p_{s-1,\delta+1}$ implies c, u and k vanish provided σ is close to σ. This means, in view of (4.13) and (4.14), that $\mathscr{L}'(\sigma)$ will vanish on all tangent vectors to the slice $\mathscr{S}$, which will immediately give us that there are no C^1 curves of solutions in $\mathscr{L}^{-1}(0)\cap\mathscr{S}$.

First we will show this on the spherical background σ, then we will apply Nirenberg and Walker's theorem 3.14 to extend it to neighboring solutions.

(We will continue to write u when it would be more correct to use $\hat{u}$, keeping in mind that they are determined by each other through (4.9) and that u is less differentiable across the star boundary. Also, in the case of equations of state which give $s=3$ we will regard functions such as $dM/d\bar{U}$ as piecewise C^0 functions, not as distributions.)

The splitting of $T_\sigma \mathcal{P}^p_{s-1,\delta+1} = T_\sigma \mathcal{O}_\sigma \oplus T_\sigma \mathcal{S}$ allows (4.16) to be investigated on the two subspaces separately.

Proposition 4.2: The equation (4.16) has no nonzero solution (c,u,k) on $T_\sigma \mathcal{O}_\sigma$.

Proof: Since $\mathcal{L}(\varphi^* \sigma) = \varphi^* \mathcal{L}(\sigma) = 0$ for any $\varphi \in \mathcal{D}^p_{s,\delta}$ it is clear from (4.13) that any $(c,u,k) \in T_\sigma \mathcal{O}_\sigma$ is a solution of (4.16) iff $K_\gamma \circ \mathrm{div}_\gamma \circ K_\gamma \xi = 0$ since $c = K_\gamma \xi$ for a unique $\xi \in X^p_{s,\delta}$. But since there are no Killing vector fields on $\mathbb{R}^3$ that vanish at infinity (Choquet and Choquet-Bruhat (1978)) it follows that $\mathrm{div}_\gamma \circ K_\gamma \xi = 0$ which implies $\xi = 0$ by corollary 3.9. Thus $(c,u) \in T_\sigma \mathcal{O}_\sigma$ is zero, whence $L(\sigma)(c,u) = 0$, so $k=0$. ∎

Unfortunately, on $T_\sigma \mathcal{S}$ the situation is considerably more complicated. We are unable to solve equation (4.16) directly, even with the extra condition $\mathrm{div}_\gamma c = 0$. One approach used to study second order elliptic differential equations on asymptotically Euclidean manifolds is to make a conformal transformation to a compact manifold so that theorems known in the compact case can be used (Choquet-Bruhat (1977)). However, there seems to be no such compactification of Σ which allows the coefficients of the "compactified" equations corresponding to (4.16) to be non-singular. Even when equation (4.16) is written in the (2+1)-dimensional formalism and a spherical harmonic expansion is done, the spherical ($\ell=0$) portion of the equations results in several coupled second order equations which can not be solved explicitly even in vacuo and which seem to be unamenable to a maximum principle argument except in the case $k=0$, where it can then be shown that all the spherical variations vanish.

On the other hand, Künzle (1971) has linearized the static field equations (1.59-1.71) in the (2+1)-dimensional formalism with the requirement that the central value of the potential U_c and the pressure p_c remain constant. This means in our present situation that $k=0$ since it follows from (2.9) that

$$k=\delta U_c+(\rho_c+p_c)^{-1}\delta p_c. \qquad (4.17)$$

These linearized equations turned out to decouple into a three dimensional scalar second order equation which could be solved by maximum principle arguments and into equations on the compact equipotential surfaces which could be solved using a lemma which is valid for Riemannian 2-manifolds with non-negative curvature. (An error in the proof of this lemma given in Künzle (1971) is corrected in appendix 1.) The maximum principle arguments which Künzle gave used the assumption that $\rho \geq 3p$ but the extension to the case $\rho \geq p$ is straightforward and is presented in appendix 2 along with the linearization arguments of Künzle which we use a little later.

We will therefore reduce the present problem to the one treated by Künzle by showing that $\delta m=0$ implies that $\delta U_c=0=\delta p_c$ or, equivalently in view of (4.17), that $k=0$ and $\Psi_c=0$ since $\delta U_c=\Psi_c+\mathcal{L}_\xi U|_{U=U_c}=\Psi_c$. To this end one could derive (for $(c,u,k)\in T_\sigma\mathcal{S}$ so that $\mathrm{div}_\gamma c=0$) from (4.16) three coupled second order scalar equations involving u,

$$\Phi:=c^r_r \text{ and } \psi:=\Phi-W^{-1}c_{ij}U^iU^j. \qquad (4.18)$$

Since only spherical variations need not vanish at the center and the general variation (on the spherical background) is obtained by superposition of spherical harmonics we need only look at the $\ell=0$ component, i.e. we can confine ourselves to purely spherical variations. However, as we noted above, this system is still too complicated and we must proceed differently to make some progress.

As in §2.3, with a suitable choice of coordinates the Einstein equations for a spherically symmetric static spacetime reduce to just two first order ordinary differential equations for two quantities which we can, for example, choose to be (2.2) and (2.3). In these equations r denotes the curvature radius of the equipotential surfaces. We now consider a one-parameter family of spherical perfect fluid spacetimes described in terms of a fixed radial coordinate r'. We denote by $r=F(\lambda,r')$ the curvature radius, where λ is the parameter of the curve in $\mathcal{P}^{p}_{s-1,\delta+1}$ whose tangent is the (spherical) variation (c,u,k) at one fixed solution, given by $\lambda=0$, and we choose $F(0,r')=r'$ and write $\partial F/\partial\lambda|_{\lambda=0}=:f(r')$. The coordinate freedom is again eliminated as previously: We want the variations to take the form

$$\delta U=\Psi+\mathcal{L}_{\zeta}U, \quad \delta\gamma_{ij}=c_{ij}=\Phi_{ij}+\mathcal{L}_{\zeta}\gamma_{ij} \tag{4.19}$$

where again $\nabla^{i}\Phi_{ij}=0$, but now $\zeta=f(dr'/dr)d/dr'$.

Writing (2.2) and (2.3) in terms of the polar coordinate r' we get

$$dW^{2}/dr'=(dF/dr')(2MWA^{-1}-4W^{2}F^{-1}), \tag{4.20}$$

$$dU/dr'=(dF/dr')WA^{-1}, \tag{4.21}$$

where A is defined through (2.61). Linearizing these and using (4.12), it turns out that f drops out and we find (after some manipulations using $\nabla^{i}\Phi_{ij}=0$), for $\lambda=0$ where $r'=r$,

$$W^{2}r^{2}d\Psi/dU-Mr^{2}\Psi+\tfrac{1}{2}(1+\tilde{p}r^{2})\phi=-kr^{2}(M-\tilde{p}) \tag{4.22}$$

and

$$W^{2}r(dr/dU)d\phi/dU+2(1+2\tilde{p}r^{2})\phi-8Mr^{2}\Psi=-8kr^{2}(M-\tilde{p}) \tag{4.23}$$

where $\phi := W^{-2}\Phi_{ij}U^iU^j$. These equations are much simpler than those that result from (4.16) for spherical variations since f drops out before a Lie derivative of ΔU has to be subtracted. This results because here we are fixing the coordinate system in such a way that r remains the curvature radius. In (4.16) the coordinate r is allowed to vary from the curvature radius and then this coordinate change is factored out.

Lemma 4.3: If for variations $(\delta\gamma_{ij},\delta U,\delta U_b)$ on the spherical background the variation of the mass vanishes, $\delta m=0$, then also δU_c, δp_c and $\delta U_b=k$ vanish.

Proof: Equations (4.22) and (4.23) can be explicitly integrated in the vacuum region using the vacuum solution (2.4,2.5) and give

$$\Psi = m^{-1}\delta m \tanh U \text{ and } \phi = -2m^{-1}\delta m \tanh^2 U \qquad (4.24)$$

where one integration constant was put equal to zero in view of the asymptotic conditions and the other identified with the mass change δm. Requiring Ψ to be in $M^p_{s-2,\delta+1}$, however, forces δm to vanish, since $\tanh U = m/r + O(r^{-2})$ for $r \longrightarrow \infty$. Since $\phi\gamma_{ij}U^iU^j = \Phi_{ij}U^iU^j$, $\phi\gamma_{ij}$ clearly has the same differentiability properties as Φ_{ij} so $\phi \in M^p_{s-1,\delta+1}$. (Recall that we are on the spherical background so $\gamma_{ij} \in \mathcal{R}^p_{s,\delta+1}$.) It is easily seen that

$\mathfrak{p}r^2 \in M^p_{s-2,\delta+2}$ and from (2.3) that $W(dr/dU) \in M^p_{s-2,\delta}(1)$ so that all the terms in (4.23) are in $M^p_{s-2,\delta+1}$ and thus in C^{s-3} except the term $-8Mr^2(\Psi - k)$. But $M \notin M^p_{s-2,\delta+2}$ because of its behaviour at the star boundary while all derivatives of Ψ up to order $s-3$ vanish at the star boundary so we must have $k=0$. By our assumption (iv) of §2.4 on the differentiability of ρ in the interior region, M is more differentiable in the interior so we see that $Mr^2\Psi \in M^p_{s-2,\delta+2}$ and equation (4.22) then implies $d\Psi/dU \in M^p_{s-2,\delta+1}$ so $\Psi \in M^p_{s-1,\delta+1} \subset C^{s-2}$. Since $s \geq 3$, Ψ is at least C^1.

In the interior region, then, (4.22) and (4.23) form a regular linear homogeneous system with continuous coefficients in $(U_c, U_b]$ with zero initial conditions at U_b so that $\phi = \Psi = 0$ identically (c.f. Hartman (1973)) and therefore, in particular, at the center. Equation (4.17) and the fact that $\partial_i U(x_c) = 0$ now imply that δU_c and δp_c vanish. ∎

In order to use Künzle's (1971) result the relation of the linearizations in the 3-dimensional and the (2+1)-dimensional formalisms must be determined. Since $U \in M^p_{s-1,\delta}$ has only one critical point that is nondegenerate the same is true for U sufficiently close to $U \in M^p_{s-1,\delta} \subset C^1$. Thus any (U, γ) sufficiently close to (U, γ) in $\mathcal{P}^p_{s-1,\delta+1}$ can be described in terms of $(\mathbf{W}, \boldsymbol{\gamma})$ in the (2+1)-dimensional formalism of §1.4 using coordinates $(\mathbf{x}^i) = (U, \mathbf{x}^A)$. (Recall that $\boldsymbol{\gamma}$ is the induced metric on the

U=const. 2-spheres.)

Now, $W \in M^{p}_{s-2,\delta+2}$ and $\bar{\gamma}_{AB}(U) \in M^{p}_{s-1,\delta+1}(\mathring{\gamma}_{AB})$, $\mathring{\gamma}_{AB}$ being the metric of the standard unit sphere. Now any C^1 curve through σ with its tangent in $T_\sigma \mathscr{P}^{p}_{s-1,\delta+1}$ can also be represented by a curve in $\mathcal{M}^{p}_{s-1,\delta+1} := M^{p}_{s-2,\delta+2} \oplus M^{p}_{s-1,\delta+1}(\mathring{\gamma}_{AB})$ through $(W,\bar{\gamma})$ whose tangent at $(W,\bar{\gamma})$ is in $M^{p}_{s-2,\delta+2} \oplus \bar{S}^{p}_{s-1,\delta+1}$. Let $(\delta\gamma, \delta\hat{U}, 0) \in T_\sigma P^{p}_{2,\delta+1}$ and $(w,\bar{\tau}) \in T_{(W,\bar{\gamma})}\mathcal{M}^{p}_{s-1,\delta+1}$. Noting that

$$U=\bar{x}^1,\ \gamma_{ij}=(\partial_i \bar{x}^1)(\partial_j \bar{x}^1)W^{-2}+(\partial \bar{x}^K/\partial x^i)(\partial \bar{x}^L/\partial x^j)\bar{\gamma}_{KL}$$

and linearizing we find that $\delta U=\delta\bar{x}^1$ and that

$$\delta\gamma_{ij}=2W^{-2}U_{(i}\partial_{j)}\delta\bar{x}^1-2W^{-3}U_iU_j\delta W+2\bar{\gamma}_{KL}\partial_{(i}\bar{x}^K\partial_{j)}\delta\bar{x}^L$$
$$+\partial_i\bar{x}^K\partial_j\bar{x}^L\delta\bar{\gamma}_{KL}$$

where δW and $\delta\bar{\gamma}_{KL}$ contain variations due to the change in the coordinate system $(U,\bar{x}^A)$ along the curve in $\mathcal{M}^{p}_{s-1,\delta+1}$ as well as the tangent to the curve in $\mathcal{M}^{p}_{s-1,\delta+1}$ ($\delta\bar{\gamma}_{KL}$ is not a tensor), namely

$$\delta W=w+(\partial_U W)\Psi$$

$$\delta\bar{\gamma}_{KL}=\bar{\tau}_{KL}+W^{-1}\Omega\Psi\bar{\gamma}_{KL}+2\bar{\Gamma}^{A}_{B(K}\bar{\gamma}_{L)A}\delta\bar{x}^B.$$

(Recall from §1.4 that $\Omega=\bar{\gamma}^{KL}\Omega_{KL}$ where Ω_{KL} is the second fundamental form of the U=const. topological spheres. Also, $\bar{\Gamma}^{A}_{BC}$ are the Christoffel symbols of the metric $\bar{\gamma}$.)

Thus, defining $\zeta:=(\delta U,\delta\bar{x}^{A})$ in the $(\bar{x}^{i})$-coordinate system so that $\delta U=\zeta^{1}=\zeta^{i}U_{i}$ and $\zeta\in X^{p}_{s,\delta}$ we have

$$\delta\gamma_{ij}=2\nabla_{(i}\zeta_{j)}+\partial_{i}\bar{x}^{K}\partial_{j}\bar{x}^{L}\bar{\tau}_{KL}-2U_{i}U_{j}W^{-3}w.$$

Since the central potential and pressure are fixed to first order and a variation of higher order will not affect the first order quantities w and $\bar{\tau}_{KL}$ the result of Künzle (1971), given in appendix 2, yields $w=0$ and $\bar{\tau}_{KL}=2\bar{\nabla}_{(K}\chi_{L)}$ for some vector field χ^{A} on the U=const. hypersurfaces.

One can choose a coordinate system $(U,\bar{x}^{A})$ such that $\bar{\gamma}_{1A}=0$ identically. Then $(\mathcal{L}_{\chi}\gamma)_{1A}=\bar{\nabla}_{(1}\chi_{A)}=0$ which implies $\partial_{U}\chi^{A}=0$. But χ^{A} must vanish at infinity since we require $(0,\chi^{A})\in X^{p}_{s,\delta}$ so $\chi^{A}=0$. Thus δU and $\delta\gamma_{ij}$ are merely Lie derivatives, hence so is $\delta\hat{U}$, and we conclude that if $(\delta\gamma,\delta\hat{U},k)$ satisfy (4.16) and are in $T_{\sigma}\mathcal{S}$ they must all vanish. Summarizing this we have

Theorem 4.4: The operator equation

$$L(\sigma)(c,u)=kq(\sigma)$$

where L and q are defined by (4.13) and (4.15), respectively, and

$$(c,u,k)\in T_\sigma \mathcal{P}^p_{s-1,\delta+1}=S^p_{s-1,\delta+1}\oplus M^p_{s-1,\delta+1}\oplus \mathfrak{R}$$

implies $(c,u,k)=0$.

4.4 Curves of solutions in $\mathcal{S}$

We now wish to extend this result to slices supposedly representing nontrivial static deformations of the spherically symmetric solution in order to show that there are really no (sufficiently regular) such deformations. It seems natural to apply Nirenberg and Walker's theorem 3.14 to get this result, but there are still a few problems. In order to apply the Berger–Ebin decomposition to the variations of the 3-metric and thus for the proof of theorem 3.13 we need a fall off rate of $1/r^2$ or higher. However, the isomorphism theorem 3.7 and Nirenberg and Walker's theorem both require a $1/r$ fall off, the former being required in order to have a closed range. Without the closed range result the presence of the inhomogeneous term would make it hard to extend theorem 4.4 to a neighborhood of σ. We must therefore now investigate the operator $L(\sigma)$ also on the larger space $S^p_{s-1,\delta}\oplus M^p_{s-1,\delta}$.

Writing $L(\sigma)=L_\infty+\sum_{|\mu|\le 2} b_\mu D^\mu$ it is readily verified from (4.11) and (4.13) that each b_μ contains derivatives of U and γ up to order $2-|\mu|$ for all $|\mu|\le 2$. The properties (3.6–3.8) of the weighted Sobolev spaces then easily show that for $\sigma\in\mathcal{P}^p_{s-1,\delta+1}$, $b_\mu\in M^p_{s-3+|\mu|,\delta+3-|\mu|}$ for all $|\mu|\le 2$. If in addition the b_μ are smooth enough, (they actually only need be continuous for $|\mu|=0$) in the asymptotic region the conditions on b_μ for theorems 3.7 and 3.14 will be satisfied. This is certainly the case for any $\sigma\in S_\rho$ since we have only compact matter distributions and the asymptotically flat solutions to the field equations are analytic in the vacuum region (Müller zum Hagen et al. (1970)). Application of theorem 3.7 gives the following.

Proposition 4.5: For $\sigma\in\mathcal{P}^p_{s-1,\delta+1}\cap S_\rho$ the map

$$L(\sigma):P:=S^p_{s-1,\delta}\oplus M^p_{s-1,\delta}\longrightarrow S^p_{s-3,\delta+2}\oplus M^p_{s-3,\delta+2}=:Q \qquad (4.25)$$

is a continuous linear operator with finite dimensional kernel and closed range.

If we try to solve

$$L(\sigma)x=kq(\sigma) \text{ for } k\in\mathbb{R},\ x\in P \qquad (4.26)$$

using the method used in the proof of lemma 4.2 we find that Ψ and ϕ for the spherical variation are uniquely determined by δm in the vacuum region. They determine therefore k on the boundary and hence the value δU_c. Thus there is a one-dimensional solution subspace of $P\oplus\mathbb{R}$, spanned by (x_0,k_0), say, where (x_0,k_0) is the solution corresponding to $\delta m=m$ in the vacuum region.

Proposition 4.6: For σ in a neighborhood of σ in $\mathcal{P}^p_{s-1,\delta+1}$ the operator $L(\sigma):P\longrightarrow Q$ is injective.

Proof: Let first $\sigma=\sigma$. Putting $k=0$ in (4.23) means $\delta U_b=0$, i.e. $(\rho_c+p_c)\delta U_c+\delta p_c=0$ and, as we just saw above, $\delta m=0$. Therefore there is no spherical solution for which the central pressure and potential stay fixed. Hence, just as in theorem 4.4 there are no nonspherical variations either. The operators $L(\sigma)$ and $L(\sigma)$ on P, however, satisfy the hypothesis of theorem 3.14 since $\|\sigma-\sigma\|_{p,s-1,\delta+1}$ small implies $\|b_\mu(\sigma)-b_\mu(\sigma)\|_{p,s-3+|\mu|,k-|\mu|}$ is small, so that the kernel of $L(\sigma)$ can have no higher dimension than $\ker L(\sigma)=\{0\}$. ∎

But, finally, we must again consider equation (4.16) for $x=(c,u)\in P_e=S^p_{s-1,\delta+1}\oplus M^p_{s-1,\delta+1}$ and k a constant to be determined. By (3.4) P_e is a subspace of P but not a closed one. On the other hand, for σ, we know that

$$x\in P_e,\ L(\sigma)x=kq(\sigma)\Longrightarrow x=0,\ k=0.$$

This means that $q(\sigma)\notin L(\sigma)P_e$ while according to the above there is a unique $x_0\in P$ such that

$$L(\sigma)x_0=q(\sigma). \tag{4.27}$$

Now suppose that σ is in a neighborhood of σ such that

$$\begin{aligned}&\|L(\sigma)-L(\sigma)\|<\varepsilon_1 \text{ (operator norm)}\\ &\text{and } \|q(\sigma)-q(\sigma)\|_{p,s-3,\delta+3}<\varepsilon_2\end{aligned} \tag{4.28}$$

for some small ε_1, $\varepsilon_2>0$. Since $\ker_P L(\sigma)=0$ there is at most one solution $x \in P$ of

$$L(\sigma)x = q(\sigma). \tag{4.29}$$

We will show that x cannot lie in P_e.

Since $L(\sigma): P \longrightarrow Q$ is injective and has closed range it has a bounded inverse whence there exists $C>0$ such that

$$\|L(\sigma)\|_{p,s-3,\delta+2} \geq C\|x\|_{p,s-1,\delta} \text{ for all } x \in P. \tag{4.30}$$

From (4.27) and (4.29) we have $q(\sigma)-q(\sigma)=L(\sigma)(x-x_0)$ $+(L(\sigma)-L(\sigma))x_0$ which leads, together with (4.28) and (4.30), to

$\|x-x_0\|_{p,s-1,\delta} \leq (C-\varepsilon_1)^{-1}\|L(\sigma)(x-x_0)\|_{p,s-3,\delta+2}$

$\leq (C-\varepsilon_1)^{-1}(\|q(\sigma)-q(\sigma)\|_{p,s-3,\delta+3}+\|L(\sigma)-L(\sigma)\|\ \|x_0\|_{p,s-1,\delta})$

$<(C-\varepsilon_1)^{-1}(\|x_0\|_{p,s-1,\delta}\varepsilon_1+\varepsilon_2)$ i.e. the solution x is arbitrarily close to x_0 in P for small enough ε_1 and ε_2. On the other hand it is not hard to see that

$$d(x_0, P_e) := \inf_{y \in P_e} \|x_0 - y\|_{p,s-1,\delta} > 0.$$

(Observe that x_0 is explicitly known in the vacuum region and has an asymptotic expansion starting with a term proportional to $1/r$, while for $y \in P_e$ the mean falloff rate is higher). Together with the remarks made after equation (4.16) this leads to the following uniqueness result for static spacetimes.

Theorem 4.7: Let $c:[0,1] \longrightarrow \mathscr{S} \cap S_{\rho(p)}$ be a C^1 curve such that $c(0)=\sigma$ is a spherical solution. Then c is constant if the slice $\mathcal{S}$ is contained in a small enough neighborhood of σ.

Proof: Since $\mathcal{S}$ is a C^1 Banach manifold and $c(t)\in S_\rho$ implies that $\mathcal{L}(c(t))=0$ whence $\mathcal{L}'_{c(t)}(dc/dt)=0$ for all t. But since c is also tangent to the slice $\mathcal{S}$ this is equivalent to $L(c(t))(dc/dt)=kq(c(t))=0$ which implies $dc/dt=0$ for all t. ■

CHAPTER V

ON THE UNIQUENESS PROBLEM FOR SLOWLY ROTATING STATIONARY FLUIDS

5.1 Introduction

In this chapter the advantage of the Banach space formalism developed for the static case will become apparent as it can be applied also to the stationary case resulting in a similar type of uniqueness result. Namely, there are no nonconstant differentiable (C^1) curves of rigidly rotating, perfect fluid solutions of Einstein's equations with fixed mass m, fixed surface temperature T_b, fixed equation of state $\rho(p)$ and fixed angular momentum J which are close to the spherically symmetric solution, i.e. which are slowly rotating so that J is small. This result was presented for equations of state which result in a discontinuous matter density at the boundary of the star by Künzle and Savage (1980c) using results obtained from the linearized (2+1)-dimensional formalism of Einstein's equations for stationary spacetimes which was given in Künzle and Savage (1980a). This latter paper contains an analogous investigation for the stationary case as was carried out by Künzle (1971) for the static case. In other words, it is a linearization keeping the central potential U_c and pressure p_c fixed instead of the more physical constraint of fixing the mass. We will be able to use the relation between these two linearizations which was obtained in the last chapter to obtain our result.

The extension to more general equations of state which satisfy conditions (i) to (v) of §2.4 follows easily as in the static case but one additional restriction must be made for mathematical reasons. In analysing the linearized equations we need to make a power series expansion in the interior region so we must also assume that the equation of state is analytic. We will assume then that $\tilde{S}_{\rho(p)}$ denctes the set of asymptotically Euclidean stationary axisymmetric solutions of Einstein's equations with a fixed surface temperature T_b and a fixed analytic equation of state $\rho(p)$ satisfying conditions (i) to (v) of §2.4. To demonstrate the similarity to the static case and to avoid a proliferation of symbols we will use a tilde above a symbol used to denote a map or space in the static case to denote the corresponding map or symbol in the stationary case.

Recall that Lindblom (1976) has shown that equilibrium fluid configurations satisfying our conditions for a stellar model are necessarily axisymmetric. However, his method, which is similar to Hawking's (1972) proof that stationary black holes are axisymmetric, uses analytic extension methods which are somewhat difficult to compare rigorously with our asymptotic conditions. Nevertheless this result shows that we can restrict to the axisymmetric case without a serious loss of generality so, as in §1.3, we take a fixed vector field η on Σ which behaves asymptotically, in cartesian coordinates, like $\eta = x^1\partial_2 - x^2\partial_1 + O(|x|^{-1})$.

Thus we are eliminating some coordinate freedom. The angular momentum J is then given by (1.38) or (1.39) where α is a 1-form such that $H=d\alpha$ (c.f. §1.3).

The slicing theorem we require in the stationary case again depends on the existence of a Berger–Ebin type decomposition but it also depends on the decomposition of vector fields given by corollary 3.11. However, since we have the asymptotic properties (4.1) which give a fall off rate of $1/r^2$ for both γ and α this creates no problems. In fact, using the vacuum field equations together with the asymptotic properties of η and (4.1) it is seen that

$$h^i=2J|x|^{-3}(\delta^i_3-3|x|^{-2}x^3x^i)+O(|x|^{-4}) \tag{5.1}$$

while the vector potential α is given (up to a gradient) by

$$\alpha_i=2J|x|^{-3}\varepsilon_{ik3}x^k+O(|x|^{-3}). \tag{5.2}$$

In order to obtain an elliptic operator to which we can apply Cantor's isomorphism theorem and Nirenberg and Walker's theorem, Einstein's equations (1.22–1.24) and the thermodynamic equilibrium equations (1.25–1.28, 1.11) must be modified so that the linearized system will consist only of as many second order equations as there are unknown functions.

We use the vector potential α to do this. Equation (5.2) implies $\alpha\in X^p_{s-1,\delta+1}$ for $0\le\delta<1-3/p$ for some p and s which we will see later must satisfy $s\geq 2$ and $p>3$ so that corollary 3.11 then implies α will be unique if we require

$$\mathrm{div}_\gamma\alpha=-\nabla^i\alpha_i=0 \tag{5.3}$$

Using α instead of h, where $h^i=\epsilon^{irs}\partial_r\alpha_s$, makes equation (1.24) into a second order equation and eliminates equation (1.28) as it is then identically satisfied.

In the matter region D we have from (1.27) that $\mathcal{L}_\theta\alpha=0$ for the unique α defined by $H=d\alpha$ and (5.3) so we can integrate the rigid rotation condition equation (1.26) and find that

$$\theta\lrcorner\alpha-vT^{-1}e^{-2U}=:-a=\mathrm{const} \tag{5.4}$$

From the conditions that θ is a symmetry of the spacetime, (1.25), it follows that θ vanishes at the center, whence $v_c=e^{U_c}$ and therefore

$$a=T_c^{-1}e^{-U_c}. \tag{5.5}$$

From (5.4) we then find

$$v=Te^{2U}(a+\theta\lrcorner\alpha) \tag{5.6}$$

and, solving for T,

$$T^{-2}=e^{2U}(a+\theta\lrcorner\alpha)^2-e^{-2U}\theta^2. \tag{5.7}$$

The vector field θ is not defined in the vacuum region

$\Sigma \backslash D$, but it is an infinitesimal isometry in D. Since it is also tangent to the topological 2-spheres $U=\text{const.}$ by (1.25) and vanishes at the center it must have closed orbits and thus represents an infinitesimal rotation. It follows in the axisymmetric case that unless the spacetime is spherically symmetric θ must be proportional to the given rotational generator η, i.e.

$$\theta=b\eta, \quad b=\text{const. in } D, \tag{5.8}$$

since there are no closed 2-dimensional subgroups of the isometry group of a Riemannian 2-sphere (c.f. Kobayashi (1972) p.47). Note that b/a is essentially the angular velocity. Recall that η is globally defined on Σ and satisfies (1.36) and (1.37).

Since we assume axial symmetry it might seem reasonable to eliminate an angular variable and work in the (r,θ)-half plane. However, for global arguments this quotient space is not particularly convenient and we could not immediately apply the Banach space techniques which we used in the static case. Similarly, if we were to consider η as a fixed vector field on Σ when parameterizing the set of equilibrium configurations the structure of the group of diffeomorphisms leaving η invariant would be more complicated. So it seems best to treat η also as a variable. In view of the fact that η is a Killing vector field it satisfies the second order equation

$$\mathrm{div}_{\gamma}\circ\mathcal{L}_{\eta}\gamma=-\nabla^{i}(\nabla_{i}\eta_{j}+\nabla_{j}\eta_{i})dx^{j}=0. \tag{5.9}$$

For the temperature T we have from the equation of hydrostatic equilibrium (1.11) that in D

$$T=T_b \exp\int_0^p (\rho(\overline{p})+\overline{p})^{-1} d\overline{p} \tag{5.10}$$

where T_b is the temperature on the surface ∂D of the star, defined by $p=0$. Since for each different value of T_b of the surface temperature we get an otherwise identical model with the temperature differing at each point of D by a constant factor, just as in the static case, we keep T_b fixed once and for all.

If the tensor fields γ, U, α, η and the constants a and b are given then our model will be uniquely determined since θ is given by (5.8) in the domain D, T by (5.7) and finally ρ and p as functions on D by (5.10) (or equivalently by (1.11)) since we also consider ρ as a function of p to be given. For a fixed manifold Σ which we take to be diffeomorphic to $\mathbb{R}^3$ and which thus can be considered to be provided with a fixed coordinate system, we can therefore describe the set $\tilde{\mathfrak{S}}_{\rho(p)}$ by a subset of a set $\mathfrak{P}$ of sextuples $\sigma=(\gamma,U,\alpha,\eta,a,b)$ which satisfy the appropriate differentiability conditions (determined by $\rho(p)$) and the asymptotic conditions (4.1, 1.37).

Then, as in the static case, we can consider $\tilde{\mathfrak{S}}_{\rho(p)}$ to be the inverse image of zero of a map $\mathfrak{F}:\mathfrak{P}\longrightarrow\mathfrak{Q}$ between Banach spaces.

We will take the map $\tilde{\mathscr{L}}$ to be given by

$$\begin{aligned}\tilde{\mathscr{L}}:(\gamma,U,\alpha,\eta,a,b)\longrightarrow(&R_{ij}-2U_iU_j+\tfrac{1}{2}e^{4U}h_ih_j(\rho+p)e^{-4U}T^2\theta_i\theta_j\\ &+(2pe^{-2U}+(\rho+p)T^2\theta^2e^{-4U})\gamma_{ij},\ \Delta U-\tfrac{1}{2}(\rho+3p)e^{-2U}+\tfrac{1}{2}e^{4U}h^2\\ &-(\rho+p)e^{-4U}h^2\theta^2,\ e^{-4U}\nabla^r(\nabla_{[r}\alpha_{k]}e^{4U})+(\rho+p)e^{-6U}vT\theta_k,\ \nabla^r\nabla_{(r}\eta_{k)})\end{aligned} \tag{5.11}$$

where θ, T, v, and p are defined by (5.8), (5.7), (5.6) and (5.10), respectively.

We have chosen this particular map, so as to have as many second order equations as unknown functions. It is clear that the first three components of the equations

$$\tilde{\mathscr{L}}(\sigma)=0$$

correspond to Einstein's equations (1.22–1.24) respectively, and the last to (5.9). We have not included the stronger equations (1.36) nor the gauge condition (5.3) on α since it will turn out that the chosen equations already determine the equilibrium configuration uniquely up to the two integration constants m and J and arbitrary diffeomorphisms (asymptotic to the identity) and arbitrary gradient fields added to α. This occurs because the gauge freedom of α is factored out by the slice theorem we give in the next section in the same way that the coordinate freedom is factored out, i.e. by restricting to the slice $\tilde{\mathscr{S}}$.

5.2 A Banach manifold of nearly spherical stationary spacetimes

As we saw in §2.4 the normal derivatives of M at the star boundary can be investigated using $\lim_{p\to 0}(d^iM/d(\log T)^i)$ and this has a similar behaviour to the behaviour of $\lim_{p\to 0}(d^iM/dU^i)$ in the static case. Thus, using the results of theorem 2.1 we can find that for a given equation of state such that $\lim_{p\to 0} p^{-\varepsilon}\rho = d > 0$ with $(\ell-1)/\ell < \varepsilon \leq \ell/(\ell+1)$, $\ell \geq 0$, $\varepsilon \geq 0$ M has the same differentiability properties as were determined for the static case in §4.2. Thus we again take $M \in M^p_{s-3,\delta+n}$ for a fixed $s \geq 3$ determined by the equation of state and some $p > 3$ (and less than some upper bound determined by the equation of state) and any large n. The equations (5.11) then show that we can take $U \in M^p_{s-1,\delta}$, $\alpha \in X^p_{s-1,\delta+1}$, $\gamma \in \mathcal{R}^p_{s-1,\delta+1}$ and $\eta \in X^p_{s-1,\delta}(\eta)$ where we fix $\eta = \partial_\varphi$ in terms of a polar coordinate system related to the asymptotically cartesian system in the usual way so that η has the form (1.37) for $|x| \longrightarrow \infty$ and $0 \leq \delta < 1 - 3/p$ is fixed by the asymptotic conditions.

Since the equation of state $\rho(p)$ and the surface temperature are regarded as fixed, for every value of m (in some interval $[0, m_{crit})$) there is a unique spherically symmetric model (γ, U, U_b) since U_b is also given. Since $\theta = 0$ in the static case we have $b = 0$ so the first expression in (5.11) shows that

$\gamma \in \mathcal{R}^{p}_{s,\delta+1}$. From (5.7) we see $a = T_b^{-1} e^{-U_b}$ so that giving a is equivalent to giving U_b in the static case. Thus for a given m we can describe the unique spherically symmetric solution by $\sigma = (\gamma, U, \alpha = 0, \eta, a, b = 0)$. Again, all of the components of σ have more differentiability than those of the general σ's except for U, so we must again make use of $\hat{U} = U - f_0 \tilde{p}^{2-\epsilon}$ to define the topology, where the constant f_0 is defined as in the static case in terms of the unique spherical solution. (See the discussion below equation (4.3).) Thus $\widehat{U} \in M^p_{s,\delta}$ while $\hat{U} \in M^p_{s-1,\delta}$.

In analogy to the static case we define

$$\begin{aligned} \mathfrak{P}^p_{s,\delta} = \{\sigma = (\gamma, \hat{U}, \alpha, \eta, a, b) \mid \gamma \in \mathcal{R}^p_{s-1,\delta+1}, \ \hat{U} \in M^p_{s-1,\delta+1}(\hat{U}), \\ \alpha \in X^p_{s-1,\delta+1}, \ \eta \in X^p_{s-1,\delta}(\eta), \ a, b \in \mathbb{R}\} \end{aligned} \tag{5.12}$$

where we will again use U to do the calculations but keep in mind that it is $\hat{U}$ that defines the topology. Note that $\sigma \in \mathfrak{P}^p_{s-1,\delta+1}$ and again, any two solutions $\sigma_1, \sigma_2 \in \mathfrak{P}^p_{s-1,\delta+1}$ will have the same mass, but they do not necessarily have the same angular momentum.

The action of the diffeomorphism and gauge group on $\mathfrak{P}^p_{s-1,\delta+1}$ can now be investigated. We will see that in order to obtain a slicing theorem along the same lines as in the static case it will be necessary to let $\delta = 0$ but this is the largest space, allowing the slowest fall off rate so we do not lose anything. Let

$$\mathcal{G}^p_{s,\delta}:=\mathcal{D}^p_{s,\delta}\times M^p_{s,\delta}=\{(\varphi,\chi)|\varphi\in\mathcal{D}^p_{s,\delta},\ \chi\in M^p_{s,\delta}(\mathbb{R}^3,\mathbb{R})\} \tag{5.13}$$

and define

$$A:\mathcal{G}^p_{s,\delta}\times:\tilde{\mathcal{P}}^p_{s,\delta}\longrightarrow\mathcal{D}^p_{s,\delta}$$

$$((\varphi,\chi),(\gamma,\hat{U},\alpha,\eta,a,b))\longrightarrow(\varphi^*\gamma,\hat{U}\circ\varphi,\varphi^*\alpha+d\chi,\varphi_*^{-1},a,b) \tag{5.14}$$

where φ^* denotes the pull back by the diffeomorphism φ of the appropriate covariant tensor field. Equip $\mathcal{G}^p_{s,\delta}$ with the product differentiable structure. It also carries a group structure as the semidirect product of $\mathcal{D}^p_{s,\delta}$ and the Abelian group $M^p_{s,\delta}$ with respect to the natural action of $\mathcal{D}^p_{s,\delta}$ on Σ by pull backs.

Theorem 5.1: If $p>3$ and $s>2+3/p$ (and $\delta=0$) then

(i) A defines a continuous action of $\mathcal{G}^p_{s,0}$ on $\tilde{\mathcal{P}}^p_{s-1,1}$. $A_{(\varphi,\chi)}: \sigma\longrightarrow A((\varphi,\chi),\sigma)$ is C^∞ and if $\sigma\in\tilde{\mathcal{P}}^p_{s-1+k,1}$ then $A_\sigma:(\varphi,\chi)\longrightarrow A((\varphi,\chi),\sigma)$ is C^k for $k=0$ or 1.

(ii) If $\sigma\in\tilde{\mathcal{P}}^p_{s-1+k,1}$ (for $k=0$ or 1) then the orbit $\tilde{\mathcal{O}}_\sigma:=\{A((\varphi,\chi),\sigma)|(\varphi,\chi)\in\mathcal{G}^p_{s,\delta}\}\subset\tilde{\mathcal{P}}^p_{s-1,1}$ is a C^k submanifold.

(iii) If $\sigma\in\tilde{\mathcal{P}}^p_{s,1}$ then there exists a neighborhood V of $(I,0)$ in $\mathcal{G}^p_{s,0}$ and a slice of the action, i.e. a submanifold $\tilde{\mathcal{S}}$ of $\tilde{\mathcal{P}}^p_{s-1,1}$ containing σ such that $((\varphi,\chi),\sigma')\longrightarrow A((\varphi,\chi),\sigma')$ is a homeomorphism of $V\times\tilde{\mathcal{S}}$

onto a neighborhood W of σ in $\mathscr{P}^p_{s-1,1}$ and $\tilde{\mathcal{O}}\cap\tilde{\mathscr{F}}=\{\sigma\}$.

Proof: (i) The proof is analogous to theorem 4.1 for γ and $\hat{U}$. Assume first that $\delta\in[0,1-3/p)$ as in that theorem. Since $\chi\longrightarrow d\chi$ is smooth, $(\varphi,\alpha)\longrightarrow\varphi^{\bullet}\alpha_i=\partial\varphi^k/\partial x^i(\alpha\circ\varphi)_k$ is continuous by theorem 3.5 and linear in α, so $((\varphi,\chi),\alpha)\longrightarrow\varphi^{\bullet}\alpha+d\chi$ is C^∞ in α and continuous in (φ,χ). If $\alpha\in X^p_{s,\delta+1}$ then $\partial_k\alpha\in M^p_{s-1,\delta}$ and $(\varphi,\partial_k\alpha)\longrightarrow\varphi^{\bullet}\partial_k\alpha$ is continuous. For the corresponding proof for $(\varphi,\eta)\longrightarrow\varphi_{\bullet}^{-1}\eta$ it is necessary to set $\delta=0$ to satisfy the condition $\sigma^{|\mu|-1-\delta}D^\mu\eta$ is bounded, required in theorem 3.5, because of the asymptotic behaviour of η.

(ii) A_σ is injective since there are no non-trivial isometries in $\mathscr{D}^p_{s,\delta}$ and there are no constants in $M^p_{s,\delta}$. If $k=1$ the tangent map TA_σ at $\varphi=I$, $\chi=0$ is given by

$$(\xi,f)\longrightarrow(\mathscr{L}_\xi\gamma,\mathscr{L}_\xi\hat{U},\mathscr{L}_\xi\alpha+df,\mathscr{L}_\xi\eta,0,0)\in S^p_{s-1,1}\oplus M^p_{s-1,1}\oplus X^p_{s-1,1}\oplus X^p_{s-1,0}\oplus\mathbb{R}\oplus\mathbb{R}\cong T_\sigma\mathscr{P}^p_{s-1,1}$$

where $\xi\in X^p_{s,0}$, $f\in M^p_{s,0}$. Cantor's version of the Berger–Ebin decomposition (corollary 3.12) together with the splitting given by corollary 3.11, $X^p_{s-1,1}=d(M^p_{s,0})\oplus\mathscr{I}_\gamma$ where $\mathscr{I}_\gamma=\{\omega\in X^p_{s-1,1}|\operatorname{div}_\gamma\omega=0\}$ imply that the image of $T_{(I,0)}A_\sigma$ splits so that A_σ is a C^1 imbedding.

(iii) Choose open neighborhoods V of 0 in J_γ and W of 0 in $\mathscr{I}_\gamma$ such that $\gamma+c$ is positive defenite for $c\in V$. Then, if $\sigma\in\mathscr{P}^p_{s,1}$,

$$\tilde{\mathscr{S}}=\{(\gamma+c,\hat{U}+u,\alpha+\beta,\eta+\zeta,a+A,b+B)|c\in V,\ u\in M^{p}_{s-1,1},$$

$$\beta\in W,\ \zeta\in X^{p}_{s-1,0},\ A,B\in\mathbb{R}\}$$

is clearly an imbedded submanifold of $\tilde{\mathscr{P}}^{p}_{s-1,1}$. The homeomorphism is then obtained by use of the decomposition theorems as was done in theorem 4.1. ■

In particular, the tangent space to $\tilde{\mathscr{P}}^{p}_{s-1,1}$ at $\sigma\in\tilde{\mathscr{P}}^{p}_{s,1}$ splits so for σ we have

$$T_{\sigma}\tilde{\mathscr{P}}^{p}_{s-1,1}=T_{\sigma}\tilde{\mathscr{O}}_{\sigma}+T_{\sigma}\tilde{\mathscr{S}}. \tag{5.15}$$

We conjecture that the set of solutions of Einstein's equations $\tilde{S}_{\rho}$ in some neighborhood V of σ in $\tilde{\mathscr{P}}^{p}_{s-1,1}$, namely $\mathscr{L}^{-1}(0)\cap V$, is equal to $\bigcup_{\sigma\in\mathscr{H}}\tilde{\mathscr{O}}_{\sigma}\cap V$ where $\mathscr{H}$ is a one-dimensional submanifold of $\tilde{\mathscr{P}}^{p}_{s-1,1}$ passing through σ and parameterized by the angular momentum. As in the static case, however, $\tilde{\mathscr{L}}'(\sigma)=T_{\sigma}\tilde{\mathscr{L}}$ is not surjective at $\sigma=\sigma$ so we are unable to show that $\tilde{\mathscr{L}}^{-1}(0)$ is a submanifold. Thus we cannot show that there exist solutions with nonzero angular momentum nor obtain as strong a uniqueness condition as we would like. But since, in a neighborhood of σ, the set of equivalence classes of $\tilde{\mathscr{P}}^{p}_{s-1,1}$ with respect to the action of $\mathscr{G}^{p}_{s,0}$ is in one-to-one correspondence with the slice $\tilde{\mathscr{S}}$ through σ we show that there can be no non-constant C^{1} curve in $\tilde{\mathscr{S}}\cap\tilde{S}_{\rho}$ passing through a particular solution (existence assumed) with some angular momentum unless the curve passes through solutions with different angular momenta. This will be done by a similar procedure to that used in the static case, solving the linearized equations on the spherical background and extending the result to a neighborhood of σ by the use of Cantor's isomorphism theorem and Nirenberg and Walker's theorem. Since Nirenberg and Walker's theorem is proven by contradiction one can not determine the size of a neighborhood for which this extension is valid and therefore there is little hope of obtaining an estimate for a lower bound for the value of J which leads to a bifurcation of the solution space such as one has in the classical Newtonian series when the Maclaurin sequence bifurcates to the Jacobi ellipsoids.

5.3 Linearization of $\mathfrak{L}'$ to find $\ker \mathfrak{L}'(\sigma)$

Let $(c=\delta\gamma, \hat{u}=\delta\hat{U}, \beta=\delta\alpha, \zeta=\delta\eta, A=\delta a, B=\delta b) \in T_\sigma \mathfrak{P}^p_{s-1,1}$. Let $u=\delta U$ and recall that u and $\hat{u}$ are related through equation (4.9) with u being less differentiable than $\hat{u}$. Again we do all calculations with u keeping this in mind. The variation of the pressure can be seen to be given by

$$\delta p=(\rho+p)[-T(1+2e^{-2U}T^2\theta^2)u+\tfrac{1}{2}T^3e^{-2U}\theta^r\theta^s\Phi_{rs}-T^2\upsilon\beta_r\theta^r$$
$$-T^2(\upsilon\alpha_r-Te^{-2U}\theta_r)(B\eta^r+b\zeta^r)-T^2\upsilon A]$$

For $\sigma=\sigma\in\mathfrak{P}^p_{s,1}$ we have from the proof of theorem 5.1 the decomposition

$$c=\mathcal{L}_\xi\gamma+\Phi,\ \mathrm{div}_\gamma(\Phi)=0 \tag{5.16}$$

$$u=\mathcal{L}_\xi U+\Psi \tag{5.17}$$

$$\beta=\mathcal{L}_\xi\alpha+df+\omega,\ \mathrm{div}_\gamma(\omega)=0 \tag{5.18}$$

$$\zeta=\mathcal{L}_\xi\eta+\nu \tag{5.19}$$

where $\xi\in X^p_{s,0}$, $\Phi\in S^p_{s-1,1}$, $\Psi\in M^p_{s-1,1}$, $\omega\in X^p_{s-1,1}$, $f\in M^p_{s,0}$, $\nu\in X^p_{s-1,0}$ are all unique. Another lengthy but standard calculation gives that

$$\mathfrak{L}'(\sigma):T_\sigma\mathfrak{P}^p_{s-1,\delta+1}\longrightarrow S^p_{s-3,3}\oplus M^p_{s-3,3}\oplus X^p_{s-3,3}\oplus X^p_{s-3,2}=:\mathfrak{Y} \tag{5.20}$$

is given by

$$\begin{aligned}
&\mathfrak{L}'_1(\sigma)(c,u,\beta,\zeta,A,B)_{ij}=\\
&\tfrac{1}{2}\nabla^r\nabla_r c_{ij}+\tfrac{1}{2}\nabla_i\nabla_j c^r_r-\nabla_{(i}(\nabla^r c_{j)r})+[-3R^r_{(i}\delta^s_{j)}+R^{rs}\gamma_{ij}+\tfrac{1}{2}R\delta^r_i\delta^s_j\\
&+(R_{ij}-\tfrac{1}{2}R\gamma_{ij}-\tfrac{1}{2}h_ih_je^{4U})\gamma^{rs}+e^{4U}h^rh_{(i}\delta^s_{j)}+2(\rho+p)e^{-4U}T^2\theta^r\theta_{(i}\delta^s_{j)}-2pe^{-2U}\delta^r_i\delta^s_j\\
&-(\rho+p)T^2\theta^2e^{-4U}\delta^r_i\delta^s_j-\tfrac{1}{2}e^{-6U}(\rho+p)T^2\theta^r\theta^s(4e^{2U}\gamma_{ij}+T^2(\rho'+3)(\theta^2\gamma_{ij}-\theta_i\theta_j))]c_{rs}\\
&+4U_{(i}\partial_{j)}u+[2h_ih_je^{4U}+2(\rho+3p)e^{-2U}\gamma_{ij}+(\rho+p)e^{-4U}T^2(8\theta^2\gamma_{ij}-4\theta_i\theta_j\\
&+\tau(\rho'+3)(\theta^2\gamma_{ij}-\theta_i\theta_j))]u+h_{(i}\varepsilon_{j)}{}^{rs}e^{4U}\nabla_r\beta_s+\upsilon T(\rho+p)e^{-4U}[2e^{2U}\gamma_{ij}\\
&+T^2(\rho'+3)(\theta^2\gamma_{ij}-\theta_i\theta_j)]\theta^r\beta_r+(\rho+p)e^{-4U}T^2[2\gamma_{k(i}\theta_{j)}+2(\upsilon T^{-1}e^{2U}\alpha_k-2\theta_k)\gamma_{ij}\\
&+(\rho'+3)(\upsilon T\alpha_k-T^2e^{-2U}\theta_k)(\theta^2\gamma_{ij}-\theta_i\theta_j)](B\eta^k+b\zeta^k)\\
&+\upsilon T^2e^{-4U}(\rho+p)[2e^{2U}T^{-1}\gamma_{ij}+(\rho'+3)T(\theta^2-\theta_i\theta_j)]A,
\end{aligned} \tag{5.21}$$

$$\tilde{\mathcal{L}}'_2(\sigma)(c,u,\beta,\zeta,A,B)=$$

$$\Delta u+[2h^2e^{4U}+(\rho+3p)e^{-2U}+(\rho+p)e^{-2U}(4T^2\theta^2e^{-2U}+\tfrac{1}{2}(\rho'+3)\tau^2)]u$$

$$-U^r(\nabla^s c_{sr}-\tfrac{1}{2}\nabla_r c^s_s)+[\tfrac{1}{2}h^r h^s e^{4U}-\nabla^r\nabla^s U-\tfrac{1}{2}h^2\gamma^{rs}$$

$$-(\rho+p)T^2e^{-4U}(1+\tfrac{1}{4}(\rho'+3)\tau)\theta^r\theta^s]c_{rs}+H^{rs}e^{4U}\nabla_r\beta_s+\tfrac{1}{2}(\rho+p)(\rho'+3)\tau T\upsilon e^{-2U}\theta^r\beta_r$$

$$+(\rho+p)e^{-2U}T[-2Te^{-2U}\theta_r+\tfrac{1}{2}(\rho'+3)(\upsilon\alpha_r-e^{-2U}T\theta_r)\tau](B\eta^r+b\zeta^r)$$

$$+\tfrac{1}{2}(\rho+p)(\rho'+3)T\upsilon\tau e^{-2U}A, \tag{5.22}$$

$$\tilde{\mathcal{L}}'_3(\sigma)(c,u,\beta,\zeta,A,B)_i=$$

$$\tfrac{1}{2}\nabla^r\nabla_r\beta_i-\tfrac{1}{2}\nabla_i(\nabla^r\beta_r)+4U^r\nabla_{[r}\beta_{i]}-[\tfrac{1}{2}R^r_i+(\rho+p)e^{-6U}T^2(\upsilon^2(\rho'+3)-e^{2U})\theta_i\theta^r]\beta_r$$

$$-[H^{(k}_i\gamma^{r)s}+\tfrac{1}{2}H^{kr}\delta^s_i]\nabla_k c_{rs}+[-\tfrac{1}{2}e^{-4U}\nabla^r(H^s{}_i e^{4U})$$

$$+(\rho+p)\upsilon Te^{-6U}(\delta^r_i\theta^s+\tfrac{1}{2}e^{-2U}(\rho'+3)T^2\theta^r\theta^s\theta_i)]c_{rs}-2H_i{}^r\partial_r u$$

$$-(\rho+p)\upsilon Te^{-6U}(4+\tau(\rho'+3)\theta_i u+(\rho+p)e^{-6U}T^2[\upsilon T^{-1}\gamma_{ir}+e^{2U}\alpha_r\theta_i$$

$$+\upsilon(\rho'+3)(Te^{-2U}\theta_r-\upsilon\alpha_r)\theta_i](B\eta^r+b\zeta^r)-(\rho+p)e^{-6U}T^2((\rho'+3)\upsilon^2-e^{2U})\theta_i A, \tag{5.23}$$

$$\tilde{\mathcal{L}}'_4(\sigma)(c,u,\beta,\zeta,A,B)_i=$$

$$\nabla^r\nabla_{(r}\zeta_{i)}+\tfrac{1}{2}\eta^r\nabla_r(\nabla_s c^s_i)-\tfrac{1}{2}\nabla^r\eta_i(\nabla_s c^s_r)+\nabla^{(r}\eta^{s)}(\nabla_r c_{si}-\tfrac{1}{2}\nabla_i c_{rs})+\tfrac{1}{4}(\nabla_i\eta^r+\nabla^r\eta_i)\nabla_r c^s_s$$

$$+(\tfrac{1}{2}\nabla^s\nabla_s\eta^r+\tfrac{1}{2}R^r_s\eta^s)c_{ir}-\tfrac{1}{2}(\nabla^r\nabla^s\eta_i+R^{r\ s}_{\ i\ k}\eta^k)c_{rs}. \tag{5.24}$$

where $\rho':=d\rho/dp$ and $\tau:=1+2e^{-2U}T^2\theta^2$.

As in the static case $\tilde{\mathcal{L}}'(\sigma)$ is not elliptic but by restricting

$$\tilde{\mathscr{L}}'(\sigma)(c,u,\beta,\zeta,A,B)=0 \tag{5.25}$$

to the slice $\tilde{\mathscr{S}}$ we can obtain an elliptic operator to which we will be able to apply Cantor's isomorphism and Nirenberg and Walker's theorems. To this end define (for $x:=(c,u,\beta,\zeta)$)

$$\begin{aligned}\tilde{\mathfrak{L}}_1(\sigma)(x)_{ij}:=&\tilde{\mathscr{L}}'_1(\sigma)(c,u,\beta,\zeta,A,B)_{ij}+\tfrac{1}{2}(K_\gamma\circ\mathrm{div}_\gamma c)_{ij}\\&-vT^2e^{-4U}(\rho+p)[2e^{2U}T^{-1}\gamma_{ij}+(\rho'+3)T(\theta^2\gamma_{ij}-\theta_i\theta_j)]A\\&-(\rho+p)e^{-4U}T^2[2\gamma_{k(i}\theta_{j)}+2(vT^{-1}e^{2U}\alpha_k-2\theta_k)\gamma_{ij}\\&+(\rho'+3)(vT\alpha_k-e^{-2U}T^2\theta_k)(\theta^2\gamma_{ij}-\theta_i\theta_j)]\eta^kB,\end{aligned} \tag{5.26}$$

$$\begin{aligned}\tilde{\mathfrak{L}}_2(\sigma)(x):=&\tilde{\mathscr{L}}'_2(\sigma)(c,u,\beta,\zeta,A,B)+U^r(\mathrm{div}_\gamma c)_r\\&-\tfrac{1}{2}(\rho+p)(\rho'+3)Tv\tau e^{-2U}A\\&-(\rho+p)e^{-2U}T[-2Te^{-2U}\theta_r+\tfrac{1}{2}(\rho'+3)(v\alpha_r-e^{-2U}T\theta_r)\tau]\eta^rB,\end{aligned} \tag{5.27}$$

$$\begin{aligned}\tilde{\mathfrak{L}}_3(\sigma)(x)_i:=&\tilde{\mathscr{L}}'_3(\sigma)(c,u,\beta,\zeta,A,B)_i+\tfrac{1}{2}\partial_i(\mathrm{div}_\gamma\beta)\\&+(\rho+p)e^{-6U}T^2[(\rho'+3)v^2-e^{2U}]\theta_iA\\&-(\rho+p)e^{-6U}T^2[vT^{-1}\gamma_{ir}+e^{2U}\alpha_r\theta_i+\\&v(\rho'+3)(Te^{-2U}\theta_r-v\alpha_r)\theta_i]\eta^rB,\end{aligned} \tag{5.28}$$

$$\tilde{\mathfrak{L}}_4(\sigma)(x)_i:=\tilde{\mathscr{L}}'_4(\sigma)(c,u,\beta,\zeta,A,B)_i-\tfrac{1}{2}\eta^r\nabla_r(\mathrm{div}_\gamma c)_i. \tag{5.29}$$

It is easy to verify that $\tilde{\mathfrak{L}}(\sigma)=(\tilde{\mathfrak{L}}_1,\tilde{\mathfrak{L}}_2,\tilde{\mathfrak{L}}_3,\tilde{\mathfrak{L}}_4)(\sigma)$ is a second order elliptic operator. Equation (5.25) is now equivalent to

$$\tilde{L}_1(\sigma)(x)-\tfrac{1}{2}K_\gamma\circ\mathrm{div}_\gamma c=\tilde{q}_1(\sigma)(A,B)^T \qquad (5.30)$$

$$\tilde{L}_2(\sigma)(x)-\nabla U\lrcorner\mathrm{div}_\gamma c=\tilde{q}_2(\sigma)(A,B)^T \qquad (5.31)$$

$$\tilde{L}_3(\sigma)(x)-\tfrac{1}{2}d(\mathrm{div}_\gamma\beta)=\tilde{q}_3(\sigma)(A,B)^T \qquad (5.32)$$

$$\tilde{L}_4(\sigma)(x)+\tfrac{1}{2}\eta\lrcorner d(\mathrm{div}_\gamma c)=\tilde{q}_4(\sigma)(A,B)^T \qquad (5.33)$$

where the $\tilde{q}_i(\sigma)$ are the obvious row matrices of tensor fields obtained from equations (5.26–5.30) ($\tilde{q}_4=(0,0)$) and $(A,B)^T$ is the transpose of (A,B). We also write $\tilde{q}(\sigma)$ for $(\tilde{q}_1(\sigma),\tilde{q}_2(\sigma),\tilde{q}_3(\sigma),\tilde{q}_4(\sigma))$.

We will show that for $\sigma\in\tilde{\mathcal{F}}$ near σ and $x=(c,u,\beta,\zeta)\in S^p_{s-1,1}\oplus M^p_{s-1,1}\oplus X^p_{s-1,1}\oplus X^p_{s-1,0}$ the inhomogeneous equation

$$\tilde{L}(\sigma)(x)=\tilde{q}(\sigma)(A,B)^T \qquad (5.34)$$

implies that $c=u=\beta=\zeta=A=B=0$ if also the variation δJ of the angular momentum corresponding to (c,u,β,ζ,A,B) vanishes. This will then imply that the solution of (5.25) vanishes if (c,u,β,ζ,A,B) is tangent to the slice and the corresponding $\delta J=0$.

We will do this in the same manner as in the static case, solving (5.34) first on the spherical backgrouund σ and then extending the result using the theorems 3.7 and 3.14 about elliptic operators.

On the spherical background equation (5.34) simplifies greatly, decoupling into

$$\tilde{L}_{1,2}(\sigma)(c,u)=\tilde{q}_{1,2}(\sigma)(A,0)^T \tag{5.35}$$

$$\tilde{L}_3(\sigma)(\beta,\zeta)=\tilde{q}_3(\sigma)(0,B)^T \tag{5.36}$$

$$\tilde{L}_4(\sigma)(c,\zeta)=0 \tag{5.37}$$

where (5.35) is the set of equations (4.16) obtained in the static case. Theorem 4.4 then gives us that c, u and A vanish. Putting $c=0$ in (5.37) yields the equation $\mathrm{div}_\gamma \circ K_\gamma \zeta$ which by corollary 3.9 implies that $\zeta=0$. Letting $\kappa^i=\varepsilon^{ijk}\partial_j\beta_k$ equation (5.36) then becomes

$$\varepsilon^{ijk}\partial_j(e^{4U}\kappa_k)=2(\rho+p)T^2aB\eta^i \tag{5.38}$$

with κ^i satisfying

$$\nabla_i\kappa^i=0. \tag{5.39}$$

These equations are no easier to solve in this form than (5.36) is. However, if we write these equations in the (2+1)-dimensional formalism we obtain, with $\kappa^0:=\kappa^iU_i$, $\overline{\kappa}_A=\kappa_A$, $\overline{\eta}_A=\eta_A$ (A=1,2) and D the derivative defined by (1.46), which by the choice of coordinates which makes $\gamma_{1A}=0$ can be taken to be ∂_U,

$$D\bar{\kappa}_A+4\bar{\kappa}_A-W^{-2}\partial_A\kappa^0=2W^{-1}(\rho+p)e^{-4U}T^2aB\bar{\varepsilon}_{AB}\bar{\eta}^B \tag{5.40}$$

$$\partial_{[A}\bar{\kappa}_{B]}=0 \tag{5.41}$$

$$D\kappa^0-W^{-1}\kappa^0DW-W^{-1}\bar{\kappa}^A\partial_AW+\bar{\nabla}_A\bar{\kappa}^A+W^{-1}\Omega\kappa^0=0 \tag{5.42}$$

where all the tensor fields on the U=const. surfaces are obtained from the unique spherical solution σ. As is seen in appendix 3 these equations are the same as some of the equations obtained from linearizing the field equations (1.48–1.58) in the (2+1)-dimensional formalism keeping the central potential U_c and pressure p_c fixed (which by lemma 4.3 we know is the case here as well). The analysis of (5.40–5.42) which we give below was first given by Künzle and Savage (1980a).

Since the 1-form $\bar{\kappa}_A dx^A$ is closed on the $S_c=U^{-1}(c)$ spheres by (5.41) and since the first Betti number of S^2 is zero (c.f. Goldberg (1962) p.89) $\bar{\kappa}_A dx^A$ is exact so that there exists a function $K(U,x^A)$ on S_c determined up to an additive function of U only such that $\bar{\kappa}_A=\partial_A K$. Letting $k:=\kappa^0$, equations (5.40) and (5.44) now become

$$\partial_A(DK+4K-W^{-2}k)=2T_ce^{U_c-6U}W^{-1}(\rho+p)\bar{\varepsilon}_{AB}\bar{\theta}^B \tag{5.43}$$

$$\bar{\Delta}K=-Dk-W^{-1}(2\Omega-MW^{-1})k \tag{5.44}$$

since $T=T_c e^{U_c-U}$. To solve for K and k we let $\bar{\gamma}_{AB}dx^A dx^B = r^2(U)(d\theta^2+\sin^2\theta d\varphi^2)$. Now expand into spherical harmonics,

$$K=\sum_{\ell,m} K_{\ell m}(r)Y_{\ell m}(\theta,\varphi), \quad k=\sum_{\ell,m} k_{\ell m}(r)Y_{\ell m}(\theta,\varphi)$$

where $\iint Y_{\ell m}Y_{\ell'\ell'}\sin\theta d\theta d\varphi=\delta_{\ell\ell'}\delta_{mm'}$. Let $R'=D(\log r)$. Equation (5.44) then becomes $0=r^2Dk_0+r^2W^{-1}(2\Omega-MW^{-1})k_0$ for $\ell=0$ and

$$K_{\ell m}=r^2(\ell(\ell+1))^{-1}[Dk_{\ell m}+(4R'-MW^{-2})k_{\ell m}] \quad (\ell\neq 0) \qquad (5.45)$$

Since $\eta=\partial_\varphi$ (5.43) yields $\sum_{\ell,m}(DK_{\ell m}+4K_{\ell m}-W^{-2}k_{\ell m})\partial_\theta Y_{\ell m}= 2BT_c e^{U_c-6U}W^{-1}(\rho+p)r^2\sin\theta$ and $\sum_{\ell,m}(DK_{\ell m}+4K_{\ell m}-W^{-2}k_{\ell m})\partial_\varphi Y_{\ell m}=0$ so that

$$DK_{\ell m}+4K_{\ell m}-W^{-2}k_{\ell m}=0 \text{ for } m\neq 0 \text{ or } \ell\neq 0,1 \qquad (5.46)$$

and

$$DK_1+4K_1-W^{-2}k_1=-2B(4\pi/3)^{1/2}T_c e^{U_c-6U}W^{-1}(\rho+p)r^2. \qquad (5.47)$$

Note that K_0 is an arbitrary function of U and need not be known to determine $\bar{\kappa}_A dx^A$ uniquely.

With the use of equations (1.48) and (1.53) on the spherical background we have, from the equation for k_0, $k_0=dWr^{-2}$ for $d=$constant. By doing an expansion in normal coordinates (y^i) at the center we find that $W=\frac{1}{3}M_0|y|+O(|y|^3)$ and $r=|y|+O(|y|^3)$ so that in order for k_0 to vanish at the center (since $h^0=h^i\partial_i U=0$ at the center), $c=0$, whence $k_0=0$. Equation (5.46), with the use of equation (5.45) to eliminate $K_{\ell m}$ and equations (1.48) and (1.53) on the spherical background become

$$W^2 DD\hat{k}_{\ell m}+MD\hat{k}_{\ell m}-F_\ell(U)\hat{k}_{\ell m}=0 \text{ for } m\neq 0 \text{ or } \ell\neq 0,1 \qquad (5.48)$$

where $\hat{k}_{\ell m}=rW^{-1}e^{2U}k_{\ell m}$ and

$$F_\ell(U)=W^2(5-4R')+\ell(\ell+1)r^{-2}+2M-pe^{-2U}. \qquad (5.49)$$

Since the three-dimensional Laplacian Δ in the spherically symmetric case has the form $\Delta f=W^2DDf+MDf+\bar{\Delta}f$, equation (5.48) is equivalent to

$$\Delta\hat{k}_{\ell m}=F_\ell(U)\hat{k}_{\ell m} \text{ for } m\neq 0 \text{ or } 1\neq 0,1 \qquad (5.50)$$

We now use an argument similar to that used by Künzle (1971) in the static case (c.f. appendix 2). Let $u=Wr$, $v=pr^2e^{-2U}$ and $x=Mr^2/(3u)$ so that $u,v,x\geq 0$ in the physical domain, and $ur=$const. and $v=x=0$ in vacuo. We can show that $F_\ell(U)\geq 0$ is equivalent to $\ell^2(\ell+1)^2+9u^4+36u^2x^2+v^2+2(5\ell(\ell+1)-8)u^2+12\ell(\ell+1)ux$ $-2\ell(\ell+1)v+60u^3x-26u^2v-12uxv\geq 0$. Since $p,\rho\geq 0$, we have that $0\leq v\leq 2xu$ so that the above inequality is implied by $\ell^2(\ell+1)+9u^4$ $+12u^2x^2+2(5\ell(\ell+1)-8)u^2+8u^3x+v^2\geq 0$. Since $5\ell(\ell+1)-8\geq 2$ for $\ell\geq 1$ this shows that $F_\ell(U)>0$ for $\ell\geq 1$. The asymptotic flatness conditions with Einstein's vacuum equations give us that $U=mr^{-1}+f,\ f\in M^{p}_{s-1,1}$ and $W=mr^{-2}+g,\ g\in M^{p}_{s-2,2}$ in the asymptotic region. Together with the differentiability properties $\kappa^{i}\in X^{p}_{s-1,1}$ it is then easily seen that $\hat{k}_{\ell m}\in M^{p}_{s-1,0}\subset C^{1}$. Thus $\limsup_{r\to\infty}\hat{k}_{\ell m}=0$. Also $k(U_c)=(\kappa^{i}U_{i})(U_c)=0$, so $\hat{k}_{\ell m}=0$ at the center. These boundary conditions together with (5.50) and $F_{\ell}(U)\geq 0$ for $\ell\geq 1$ yield $\hat{k}_{\ell m}=0$ for $m\neq 0$ or $\ell\neq 0,1$ since $\hat{k}_{\ell m}$ is C^{1} and we can apply a maximum argument.

Thus the only nonvanishing component is $\hat{k}_1=\hat{k}_{10}$ which, by (5.47) and (5.45), satisfies

$$W^{2}DD\hat{k}_1+MD\hat{k}_1-F_1(U)\hat{k}_1=-d_c re^{-4U}(\rho+p) \qquad (5.51)$$

where $d_c=4B\sqrt{4\pi}3^{-1/2}T_c e^{U_c}$. In vacuo one can easily show, using the exterior Schwarzschild solution, that $W=m^{-1}\sinh^{2}U$, $r=-m\sinh^{-1}U$ and $R'=-\coth U$. The only solution of (5.51) satisfying the asymptotic conditions is then found to be

$$\hat{k}=4Ce^{U}\sinh^{2}U \qquad (5.52)$$

(C is determined by B through the interior solutions discussed below.) This agrees with what one obtains by linearizing the Kerr solution on a spherical background, in this formalism. (Hartle and Thorne (1968) have shown that the empty space metric outside any slowly rotating perfect fluid agrees with the Kerr metric up to second order in the angular velocity.)

To determine the solutions in general we introduce a coordinate z defined by $e^{U}=e^{U_c}+\tau^2 z^2$ where $\tau>0$ is chosen such that $z(\text{surface of star})=1$. Note that z behaves like a radial polar coordinate near the center since U has a positive definite critical point. Equation (5.51) can then be written as

$$\nu d^2\hat{k}/dz^2+\pi d\hat{k}/dz+\iota\hat{k}+d_c t=0 \tag{5.53}$$

where $\nu=e^{2U}W^2$, $\pi=2\tau^2 z e^{2U}M-z^{-1}(e^{2U_c}-\tau^4 z^4)W^2$, $\iota=-4\tau^4 z^2(2M-\tilde{p}-4W^2R'+5W^2+2r^{-2})$ and $t=8\tau^4 z^2 e^{-2U} r(M-\tilde{p})$. Note that since $s\geq 3$, $\nu\in M^{p}_{s-2,2}\subset C^0$, $\pi,\iota,t\in M^{p}_{s-3,1}$ and $K\in M^{p}_{s-2,0}\subset C^0$. As we noted before, we assume that the equation of state is analytic in order to do the following expansions near the center in terms of z. After a lengthy calculation (and dividing by a common factor) we obtain

$$\nu=z^2+\nu_4 z^4+O(z^6) \qquad \pi=2z+\pi_3 z^3+O(z^5)$$

$$\iota=-2+O(z^2) \qquad t=t_3 z^3+t_5 z^5+O(z^7) \tag{5.54}$$

where the coefficients depend on U_c, p_c, ρ_c, $dp/d\rho(z=0)$, etc. Considering the homogeneous equation to (5.53), we see that $z=0$ is the only singularity in $[0,1]$ and it is a regular singular point, so if we make suitable power series expansions we find that the general solution of (5.53) satisfying the regularity conditions at the center is

$$\hat{k}=\mu\hat{k}_h+\lambda\hat{k}_i \tag{5.55}$$

where $\hat{k}_h$ is the solution of the homogeneous equation with $\hat{k}_h(0)=0$, $d\hat{k}_h/dz(0)=1$ and $\hat{k}_i$ the solution of (5.53) with $\hat{k}_i(0)=0$, $d\hat{k}_i/dz(0)=0$. To determine λ and μ we use the fact that $d\hat{k}/dU$ is continuous at the star boundary together with the boundary conditions $\hat{k}(1)=Ce^{-U_b}(1-e^{2U_b})^2=:C_1$, $d\hat{k}/dz(1)$ $=-2\tau^2Ce^{-2U_b}(1-e^{2U_b})(1+3e^{2U_b})=:C_2$ (where $e^{U_b}=e^{U_c}+\tau^2$. Thus λ and μ are determined by $\mu\hat{k}_h(1)+\lambda\hat{k}_i(1)=C_1$ and $\mu d\hat{k}_h/dz(1)+\lambda d\hat{k}_i/dz(1)$ $=C_2$. These equations will uniquely determine μ and λ provided $\mathcal{W}(1)\neq0$ where $\mathcal{W}(z)=\hat{k}_h d\hat{k}_i/dz-\hat{k}_i d\hat{k}_h/dz$. Now, $\mathcal{W}(z)$ obeys, by (5.53),

$$\nu d\mathcal{W}/dz+\pi\mathcal{W}+t\hat{k}_i=0 \tag{5.56}$$

with $\mathcal{W}(0)=0$.

A power series expansion gives a unique solution for (5.56), $\mathcal{W}=-z^3/5+O(z^5)$ where the factor t_3 has been absorbed into d_c which is just a fixed constant. Thus for small $z>0$, $\mathcal{W}<0$ and $d\mathcal{W}/dz<0$. Now, $\nu,t>0$ in $(0,1]$ (see (5.54)) and $\hat{k}_h(0)>0$ in $(0,1]$ since $\hat{k}_h(0)=0$, $d\hat{k}_h/dz(0)>0$ and if $z_1\in(0,1]$ is the smallest z such that $d\hat{k}_h/dz(\dot{z}_1)=0$, then $d^2\hat{k}_h/dz^2(z_1)=-(\iota/\nu)\hat{k}_h(z_1)\geq0$ since $\iota=-4\tau^4z^2F_1(U)\leq0$. Thus $\hat{k}_h$ has no maximum in $(0,1]$. Now suppose z_2 is the smallest $z\in(0,1]$ such that $\mathcal{W}(z_2)=0$. Then

$d\mathcal{W}/dz(z_2)=-t(z_2)\hat{k}_1(z_2)\nu^{-1}(z_2)<0$ so that there exists an $\varepsilon>0$ such that $\mathcal{W}(z)>0$ for $z\in(z_2-\varepsilon,z_2)$ which contradicts $\mathcal{W}<0$, $d\mathcal{W}/dz<0$ for small z. Therefore $\mathcal{W}(1)\neq 0$ and there exist unique λ and μ solving the equations giving the boundary conditions of $\hat{k}$. Thus there is a unique $\hat{k}$ determined by B with an asymptotic form given by (5.52) and vanishing identically if $B=0$. From this it is readily verified that there is a unique solution κ to (5.38) and (5.39) which must have the asymptotic form

$$\kappa^i=C\overset{3}{\kappa}{}^i r^{-3}+0(r^{-4}) \tag{5.57}$$

and which vanishes iff $B=0$ (iff $C=0$).

We can now use this to investigate the solution β of (5.36). (Recall that $\zeta=0$.) Since $T_\sigma\tilde{\mathcal{P}}^p_{s-1,1}$ splits, i.e. $T_\sigma\tilde{\mathcal{P}}^p_{s-1,1}=T_\sigma\tilde{\mathcal{O}}_\sigma\oplus T_\sigma\tilde{\mathcal{S}}$ (equation (5.15)), we can investigate β on the two subspaces separately. Since $\beta=\mathcal{L}_\xi\alpha+df$ on $T_\sigma\mathcal{O}_\sigma$ for some $\xi\in X^p_{s,0}$, $f\in M^p_{s,0}$ and $\tilde{\mathcal{L}}(A_{(\varphi,\chi)}(\sigma))=\varphi^*(\tilde{\mathcal{L}}(\sigma))$, (5.32) implies $d(\mathrm{div}_\gamma\circ df)=0$ whence $f=0$ by corollary 3.8. On $T_\sigma\tilde{\mathcal{S}}$ we have $\mathrm{div}_\gamma\beta=0$ so that β is uniquely determined by κ and hence by B. This follows from corollary 3.9 since $\kappa=0$ implies $B=0$, $d\beta=0$ so that equation (5.36) becomes $0=\nabla^r\nabla_r\beta_i-R^r_i\beta_r=-(\mathrm{div}_\gamma\circ K_\gamma\beta)_i$. From (5.57) it is seen that β has an asymptotic expansion of $0(|x|^{-2})$ at infinity and vanishes iff $B=0$.

Taking a variation of (1.39) it is easily seen that the corresponding variation in angular momentum is

$$\delta J=(1/16\pi)\lim_{|x|\longrightarrow\infty}\int_{|x|=\mathrm{const.}}\partial_{[i}\beta_{j]}\eta^{j}\gamma^{ik}d\Sigma_{k} \qquad (5.58)$$

and so will not vanish unless $\beta=B=0$. In fact κ has an asymptotic expansion

$$\kappa^{i}=2\delta J|x|^{-3}(\delta^{i}_{3}-3|x|^{-2}x^{i}x^{3})+O(|x|^{-4}). \qquad (5.59)$$

Summarizing the above, we have the following theorem.

Theorem 5.2: The operator equation

$$\tilde{\mathfrak{L}}(\sigma)(c,u,\beta,\zeta)=\tilde{\mathfrak{q}}(\sigma)(A,B)^{T}$$

with $(c,u,\beta,\zeta,A,B)\in T_{\sigma}\tilde{\mathcal{P}}^{p}_{s-1,1}$ implies that c, u, ζ, and A vanish and that β, B are uniquely determined by δJ, vanishing iff $\delta J=0$.

5.4 Curves of solutions in $\tilde{\mathcal{P}}$

Again, the presence of the inhomogeneous terms $\tilde{\mathfrak{q}}(\sigma)(A,B)^{T}$ does not allow a direct application of theorem 3.14 so we proceed analogously to the static case with the slight complication introduced by having two constants A and B. From theorem 3.7 it is easily verified that for $\sigma\in\tilde{\mathcal{P}}^{p}_{s-1,1}\cap\tilde{S}_{\rho}$ (so, in particular, for σ) the map

$$\tilde{\mathfrak{L}}(\sigma):\tilde{P}:=S^{p}_{s-1,0}\oplus M^{p}_{s-1,0}\oplus X^{p}_{s-1,0}\oplus X^{p}_{s-1,0}\longrightarrow S^{p}_{s-3,2}\oplus M^{p}_{s-3,2}\oplus X^{p}_{s-3,2}\oplus X^{p}_{s-3,2}=:\tilde{Q}$$

is a continuous linear operator with finite dimensional kernel and closed range. Just as in the static case this follows using properties (3.6–3.8) of the weighted Sobolev spaces together with the asymptotic properties of $\sigma\in\tilde{\mathfrak{S}}_\rho$.

Equation (5.35) determines c and u uniquely in terms of δm for $x\in\tilde{\mathfrak{P}}$ while (5.36) determines $\beta\in X^p_{2,0}$ uniquely in terms of δJ and c since (5.37) determines ζ uniquely in terms of c, as seen above. Thus the solution space of $\tilde{\mathfrak{L}}(\sigma)x-\tilde{\mathfrak{q}}(\sigma)(A,B)^T=0$ in $\tilde{\mathfrak{P}}\oplus\mathfrak{R}\oplus\mathfrak{R}=\{(x,A,B)\}$ is 2–dimensional and spanned by $(x_0,0,B_0)$ and $(y_0,A_0,0)$ corresponding to the solutions with $(\delta m,\delta J)=(0,J)$ and $(m,0)$ respectively.

For $\sigma=\sigma$, putting $A=0$ means $\delta m=0$ so c and u vanish, as well as ζ by (5.37). Putting $B=0$ gives $\beta=0$. For σ in some (small) neighborhood of σ in $\tilde{\mathfrak{D}}^p_{s-1,1}$ the operator $\tilde{\mathfrak{L}}(\sigma):\tilde{\mathfrak{P}}\longrightarrow\tilde{\mathfrak{Q}}$ is injective since $\tilde{\mathfrak{L}}(\sigma)$ and $\tilde{\mathfrak{L}}(\sigma)$ then satisfy the hypothesis of theorem 3.14.

Thus there are unique $x_0,y_0\in\tilde{\mathfrak{P}}$ such that

$$\tilde{\mathfrak{L}}(\sigma)x_0=\tilde{\mathfrak{q}}(\sigma)(0,1)^T \tag{5.60}$$

$$\tilde{\mathfrak{L}}(\sigma)y_0=\tilde{\mathfrak{q}}(\sigma)(1,0)^T \tag{5.61}$$

In fact $x_0=(0,0,\beta_0,0)$ where $\beta_0=0(|x|^{-2})$ at infinity while $y_0\in\tilde{\mathfrak{P}}\setminus\tilde{\mathfrak{P}}_e$ where $\tilde{\mathfrak{P}}_e:=S^p_{s-1,1}\oplus M^p_{s-1,1}\oplus X^p_{s-1,1}\oplus X^p_{s-1,0}$ is a subspace of faster fall off. Suppose σ is in a neighborhood of σ such that

$$\|\tilde{\mathfrak{L}}(\sigma)-\mathfrak{L}(\sigma)\|<\varepsilon_1 \text{ and } \|\tilde{\mathfrak{q}}(\sigma)-\mathfrak{q}(\sigma)\|_{p,s-3,\delta+3}<\varepsilon_2 \tag{5.62}$$

for some small $\varepsilon_1, \varepsilon_2>0$. Then by theorem 3.14 $\ker_p\tilde{\mathfrak{L}}(\sigma)=\{0\}$ and there are unique solutions $x,y\in\tilde{\mathfrak{P}}$ of

$$\tilde{\mathfrak{L}}(\sigma)x=\tilde{\mathfrak{q}}(\sigma)(0,1)^T \tag{5.63}$$

and

$$\tilde{\mathfrak{L}}(\sigma)y=\tilde{\mathfrak{q}}(\sigma)(1,0)^T \tag{5.64}$$

Using the fact that $\tilde{\mathfrak{L}}(\sigma):\tilde{\mathfrak{P}}\longrightarrow\tilde{\mathfrak{Q}}$ is injective and has closed range and so has a bounded inverse we can show, exactly as was done in the static case, that x is arbitrarily close to x_0 and y is arbitrarily close to y_0 for small enough ε_1 and ε_2. Therefore y cannot lie in $\mathfrak{P}$ and $x=(c,u,\beta,\zeta)$ must have $\beta=O(|x|^{-2})$ at infinity and thus corresponds to a solution with $\delta J\neq 0$.

By the remark below (5.34) we then have

Theorem 5.3: If $c:[0,1]\longrightarrow\mathfrak{S}\cap\tilde{\mathfrak{S}}_{\rho(p)}$ is a C^1 curve of solutions having all the same (small) angular momentum J then c is constant if the slice $\mathfrak{S}$ is contained in a small enough neighborhood of σ.

Proof: Exactly as in the static case.

Note that since Nirenberg and Walker's theorem 3.14 is proven by contradiction this method cannot be used to investigate how large J can become before there is a bifurcation of the solution space.

5.5 The problem of surjectivity

Whether there is a physical (non-singular) source for the Kerr solution is a long outstanding question. (See Krasinski (1978) for a recent review of investigations of this question.) More particularly, whether a rigidly rotating perfect fluid could be the source is unknown. Indeed, it has not even been shown that there exist rigidly rotating perfect fluid solutions of Einstein's equations. We would like to be able to answer this latter question by showing that $\tilde{\mathscr{L}}(\sigma)$ is surjective, for then we would have shown, by theorem 3.2, that $\tilde{\mathscr{L}}^{-1}(0)$ is a submanifold of dimension one in a neighborhood of σ. (We would then also have that $\tilde{\mathscr{L}}$ was linearization stable at σ and the uniqueness theorems for both the static and stationary cases would be stronger.)

Unfortunately, neither $\mathscr{L}'(\sigma)$ nor $\tilde{\mathscr{L}}'(\sigma)$ are surjective. We will show this only for $\tilde{\mathscr{L}}'(\sigma)$ since the argument for $\mathscr{L}'(\sigma)$ is exactly analogous and can be obtained by dropping the tildes and using equation (4.13) instead of equations (5.26–5.29) to define the map

$$\tilde{\mathscr{A}}(\sigma): S^{p}_{s-1,1} \oplus X^{p}_{s-1,1} \longrightarrow \tilde{\mathscr{D}}$$

$$(c,\beta) \longmapsto \left(\tfrac{1}{2}K_{\gamma} \circ \mathrm{div}_{\gamma} c,\ \nabla U \lrcorner \mathrm{div}_{\gamma} c,\ \tfrac{1}{2} d(\mathrm{div}_{\gamma} \beta),\ \tfrac{1}{2}\eta \lrcorner d(\mathrm{div}_{\gamma} c)\right)$$

so that

$$\tilde{\mathscr{B}}(\sigma)(c,u,\beta,\zeta,A,B):=\tilde{L}(\sigma)(c,u,\beta,\zeta)-\tilde{q}(\sigma)(A,B)^T=\tilde{\mathscr{L}}'(\sigma)(c,u,\beta,\zeta,A,B)+\tilde{\mathscr{A}}(\sigma)(c,\beta)$$

for $(c,u,\beta,\zeta,A,B)\in T_\sigma\tilde{\mathscr{P}}^p_{s-1,1}$. For $\sigma=\sigma$ we have the splitting $T_\sigma\tilde{\mathscr{P}}^p_{s-1,1}=T_\sigma\tilde{\mathscr{O}}_\sigma+T_\sigma\tilde{\mathscr{J}}$ and our above analysis has shown that $\tilde{\mathscr{B}}(\sigma)|_{T_\sigma\tilde{\mathscr{O}}_\sigma}=\tilde{\mathscr{A}}(\sigma)|_{S^p_{s-1,1}\backslash J_\gamma\bullet X^p_{s-1,1}\backslash \mathscr{J}_\gamma}$ while $\tilde{\mathscr{B}}(\sigma)|_{T_\sigma\tilde{\mathscr{J}}}=\tilde{\mathscr{L}}'(\sigma)\text{restirct}_{T_\sigma\tilde{\mathscr{J}}}$. Thus, if $\tilde{\mathscr{L}}'(\sigma)$ is surjective so is $\tilde{\mathscr{B}}(\sigma)$. Now $\ker\tilde{\mathscr{B}}(\sigma)=\{(0,0,\beta(B),0,0,B)\}$ is one dimensional so any $x_1=(\mathcal{L}_\xi\gamma,\mathcal{L}_\xi U,\mathcal{L}_\xi\alpha+d\chi,\mathcal{L}_\xi\eta,0,0)\in T_\sigma\tilde{\mathscr{O}}_\sigma$ with ξ nonzero is clearly not in $\ker\tilde{\mathscr{B}}(\sigma)$ and $\tilde{\mathscr{B}}(\sigma)x_1=y_1\neq 0$. If $\tilde{\mathscr{L}}'(\sigma)$ is surjective there exists $x_2=(\ell,u,\beta,\zeta,A,B)\in T_\sigma\tilde{\mathscr{J}}$ such that $\tilde{\mathscr{B}}(\sigma)x_2=y_1$ so $\tilde{\mathscr{B}}(\sigma)(x_1-x_2)=0$. But then $\mathcal{L}_\xi\gamma-\ell=0$ with $\text{div}_\gamma\ell=0$ which by the Berger-Ebin type decomposition implies $\ell=0$, $\xi=0$ which contradicts our assumption.

So what can be done? The basic problem is clearly that $\ker\tilde{\mathscr{L}}'(\sigma)$ is too large to allow surjectivity onto the entire space $\tilde{\mathfrak{D}}$, or from another point of view, $\tilde{\mathfrak{D}}$ is too large. But it is by no means clear how to pick a submanifold of $\tilde{\mathfrak{D}}$ of the right size. In fact $\tilde{\mathscr{P}}^p_{s-1,1}$ has too fast a fall off rate to even be able to apply Cantor's isomorphism theorem 3.7 to show that $\tilde{\mathscr{B}}(\sigma)$ and thus $\tilde{\mathscr{L}}'(\sigma)$ have closed range. However, this may not be a fundamental block as McCowen (1979) has shown that the Laplacian on flat space E^n has a finite dimensional kernel and closed range as long as $\delta\neq n-2-n/p+k$ for some $k\in\mathbb{N}$ so it may be possible to obtain a similar generalization to elliptic operators with variable coefficients. (Cantor's theorem 3.7 is a generalization of the above result for $0 \le \delta < n-2-n/p$.) McCowen is presently working on this type of generalization. Having a closed range, however, does not guarantee that one can find an appropriate submanifold of $\tilde{\mathscr{D}}$ onto which $\tilde{\mathscr{L}}'(\sigma)$ will be surjective but it does suggest there is hope.

If such a submanifold exists it is clear that there must be a splitting of the tangent space $T_0\tilde{\mathscr{D}}=\tilde{\mathscr{D}}$ (recall $\tilde{\mathscr{L}}(\sigma)=0$) which could possibly be obtained from determining the kernel of the adjoint of $\tilde{\mathscr{L}}'(\sigma)$, $\ker \tilde{\mathscr{L}}'(\sigma)^*$, since we could then write, for any $y \in \tilde{\mathscr{D}}$, $y=\tilde{\mathscr{L}}'(\sigma)x+y_2$ with $y_2 \in \ker \tilde{\mathscr{L}}'(\sigma)^*$ where x might then be uniquely determined since then $\tilde{\mathscr{L}}'(\sigma)^* y=\tilde{\mathscr{L}}'(\sigma)^*\tilde{\mathscr{L}}'(\sigma)x$ and the operator on the right hand side is formally elliptic and self adjoint and so could be expected to be invertible. However, it is a 4th order differential operator acting on $M^p_{s-3,3}$ tensor spaces with $s \ge 3$ so neither the differentiability nor the fall off are suitable for applying Cantor's isomorphism theorem (and in fact one is mapping into some "distribution" space for small s). Only for large s do we even have the use of some of the decomposition theorems, so there seems little hope of determining an appropriate submanifold in this fashion.

Another possible avenue is to make an initial coordinate restriction such as restricting to the set of metrics which are such that the Euclidean coordinates are harmonic and then hoping to find an appropriate image space on which one could show surjectivity. For metrics which are close to Euclidean this set is a subset of $\mathcal{R}^{p}_{s,\delta}$ but it is not known if this will be true far away from flat space and it is not known if this will lead to any simplification in trying to obtain a suitable Banach space onto which $\tilde{\mathcal{L}}'(\sigma)$ will be surjective.

In short, the problems of existence and obtaining stronger uniqueness theorems are still open and require much work. However, the uniqueness theorems we have shown are of the same rigor as Carter's stationary black hole uniqueness result and, since we have shown $\tilde{\mathcal{L}}'(\sigma)$ is not surjective, improvements on them will take some additional insight into the appropriate spaces to use for modelling stationary equilibrium relativistic fluids.

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APPENDIX 1: A LEMMA OF AVEZ

The lemma we prove here is used to study a differential operator on the compact topological 2-spheres $U=\text{const.}$ that arises in the investigation of the linearized (2+1)-dimensional form of Einstein's equations (c.f. appendix 2). Since all discontinuities in derivatives of the tensor fields are going to arise only across $p=\text{const.}$ surfaces which coincide in the spherical case with the $U=\text{const.}$ surfaces we can assume that the $U=\text{const.}$ surfaces are C^∞ manifolds and that the tensor fields on them are also C^∞. The latter condition could be relaxed in the proof we give below.

Let (N,g) be a 2-dimensional C^∞ Riemannian manifold, Λ_1 the fibre of 1-forms and S_2 (S_2^0) the fiber of symmetric covariant 2-tensors (with null trace) on N. The following lemma was given by Avez (1977) (c.f. Künzle (1971)).

Lemma A1.1: If $K \in C^\infty(S_2(N))$ satisfies $\nabla^B K_{BA}=0$ and $K^A_A=C=\text{const.}$ then there exists a harmonic 1-form ϕ such that $K_{AB}=-\frac{1}{2}(C-\phi^2)g_{AB}+\phi_A\phi_B$ where $\phi^2=\phi^A\phi_A$.

This lemma is equivalent to the following.

Lemma A1.2: $K \in C^\infty(S_2^0(N))$, $\nabla^B K_{BA}=0$ implies that there exists a harmonic 1-form ϕ such that

$$K_{AB}=\phi_A\phi_B-\frac{1}{2}\phi^2 g_{AB} \qquad \text{(A1.1)}$$

Proof: (c.f. Künzle (1971)) There exists a function f and a 1-form ϕ such that $K_{AB}=fg_{AB}+\phi_A\phi_B$. Clearly $f=\frac{1}{2}\phi^2$. Now $\nabla^A K_{AB}=2\phi^A\nabla_{[A}\phi_{B]}+(\nabla_A\phi^A)\phi_B=0$. Contracting with ϕ^B implies $\nabla_A\phi^A=0$ so ϕ_A is coclosed. Then $\phi^A\nabla_{[A}\phi_{B]}=0$. But locally there exists a coordinate system such that $\phi^1=0$, $\phi^2\neq0$. Hence $\nabla_{[2}\phi_{1]}=0$ or $d\phi=0$ so ϕ is closed and thus harmonic. ∎

Künzle applied lemma A1.1 to the case $N\cong S^2$ where $\phi=0$ because there are no harmonic 1-forms on the sphere. However, the proof given is not valid because ϕ, defined by (A1.1), is not differentiable at points $x\in N$ where K (or ϕ) vanish.

The study of the differential operator

$$D:C^\infty(S_2^0(N))\longrightarrow D^\infty(\Lambda_1(N)):K_{AB}\longmapsto\nabla^A K_{AB}$$

and the more restricted lemma (which is still sufficient for our purposes) which we give below was first presented by Künzle and Savage (1978).

Lemma A1.3: If N is connected and is not conformally equivalent to a complete manifold of constant negative curvature, the harmonic 1-form of lemma A1.2 is C^∞.

Proof: A connected Riemannian manifold of dimension 2 is conformally equivalent to a complete manifold of constant curvature (Wolf (1977) p.83). It is thus sufficient to prove the lemma on the surfaces of constant non-negative curvature and to show that the kernel of D is invariant under conformal transformations.

Let $f:(N,g)\longrightarrow(\tilde N,\tilde g)$ be a diffeomorphism such that $f^*(\tilde g)=\lambda^2 g$, $\lambda\in C^\infty(N,\mathbb{R})$, $\lambda(x)\neq0$ for all $x\in N$. If $K\in C^\infty(S^0_2(N))$ one finds that $\tilde K=f^{-1*}(K)\in C^\infty(S^0_2(N))$ and $(\tilde\nabla^B\tilde K_{BA})(f(x))=\lambda^{-2}(x)(\nabla^B K_{BA})(x)$ using the fact that $\dim N=2$ and $\tilde\Gamma^A_{BC}=\Gamma^A_{BC}+2\delta^A_{(B}\partial_{C)}\log\lambda$ $-g_{BC}g^{AD}\partial_D\log\lambda$. Suppose that $(\tilde N,\tilde g)$ has a constant curvature $\tilde R\geq0$. By lemma A1.2 we can write

$$\tilde K_{AB}=\tilde\phi_A\tilde\phi_B-\tfrac{1}{2}\tilde\phi^2\tilde g_{AB} \tag{A1.2}$$

where $\tilde\phi$ is a harmonic, differentiable 1-form on $\tilde N'=\{x\in\tilde N|\tilde R(x)\neq0\}$. Also, $\tilde\phi=0$ on $\tilde N\setminus\tilde N'$ and, in particular, on the boundary $\partial\tilde N'$.

On $\tilde N'$ one finds that $\tilde\Delta(\tilde\phi^2)=2(\tilde\nabla^A\tilde\phi^B\tilde\nabla_A\tilde\phi_B+\frac{1}{2}\tilde R\tilde\phi^2)\geq0$ since $\tilde R\geq0$. If $\partial\tilde N'$ is not empty, then $\tilde\phi=0$ on $\tilde N$. The decomposition (A1.2) then always holds with $\tilde\phi$ differentiable since if $\tilde K(x)=0$ for one point $x\in\tilde N$, $\tilde K(x)\neq0$ for all $x\in\tilde N$. ■

This lemma is sufficient for our purposes since we will be dealing with topological 2-spheres but let us make a few general remarks about the operator D. If $N\cong S^2$, $d:=\dim\ker D=0$ while if $N\cong T^2=S^1\times S^1$, $d=2$. (On the flat torus, $ds^2=dx^2+dy^2$, $K\in\ker D$ iff $K_{AB}=\text{const}$.)

The method of the preceding proof cannot be applied to orientable surfaces which are conformally equivalent to surfaces of constant negative curvature. However, it is easily verified that D is elliptic which suggests that lemma A1.3 remains valid for all 2-surfaces. The kernel of D could be investigated as follows. Using the scalar products $(K,L):=\frac{1}{2}\int K_{AB}L^{AB}\mathrm{vol}_g$, $K,L\in C^\infty(S_2^0)$ and $(\phi,\psi):=\int\phi^A\psi_A\mathrm{vol}_g$, $\phi,\psi\in C^\infty(\Lambda_1)$ the adjoint operator is found to be

$$D^*:C^\infty(\Lambda_1)\longrightarrow C^\infty(S_2^0):\psi_A\longmapsto -2\nabla_{(A}\psi_{B)}-\nabla^C\psi_C g_{AB}.$$

Thus $\ker D^*$ consists of the 1-forms ψ for which $\psi^A\partial_A$ is the generator of the group of conformal transformations (the group of infinitesimal coordinate changes which leaves the conformal metric $\tilde{g}_{AB}=g^{-1/2}g_{AB}$, which is independent of arbitrary scale changes in g, invariant). By the Atiyah–Singer index theorem (c.f. Palais (1965)) the analytic index $i=\dim\ker D-\dim\ker D^*$ is a topological invariant which can be calculated from the topological properties. One finds that $i=-6$ for the sphere (the group of conformal transformations is $SL(2,\mathbb{C})$) and $i=0$ for the torus but unfortunately, for other compact surfaces neither $\ker D$ nor the conformal group nor the topological index are easy to determine.

APPENDIX 2: SOLUTION OF THE LINEARIZED, (2+1)–DIMENSIONAL STATIC FIELD EQUATIONS ON THE SPHERICAL BACKGROUND

The following result was first given by Künzle (1971) with the slight exceptions that he assumed $0 \leq 3p \leq \rho$ and that $P_A = \gamma_{1A}$ was not necessarily zero while below we assume that $0 \leq p \leq \rho$ and, for a slight gain in simplicity with no loss in generality (c.f. §1.4), we use only coordinates for which $\gamma_{1A} = 0$. From §1.4 it is evident that we can regard a static, perfect fluid spacetime with a fixed equation of state, fixed central potential U_c and fixed central pressure p_c as characterized by the set $\mathcal{E} = \{(\bar{\gamma}, \Omega, W, p)\}$ where all these tensors are functions of $U \in (U_c, 0)$ and where we will drop the bars for the rest of this appendix since we will always be dealing with tensors in the (2+1)–dimensional formalism.

In order to solve the linearized field equations on the spherical background, consider a 1–parameter family $\mathcal{E}(\lambda)$ where $\mathcal{E}(0)$ is the spherically symmetric solution and follow the linearization procedure as described in §3.1. We will factor out coordinate transformations in the same manner as was done for the three dimensional case. Specifically, we write $\delta\gamma_{AB} =: C_{AB}$, $\delta W = w$. On the compact Riemannian manifold $S_c = U^{-1}(c)$ there is a Berger–Ebin (1969) decomposition

$$C_{AB} = \phi_{AB} \mathcal{L}_{\xi} \gamma_{AB}, \quad \nabla_B \phi^B_A = 0 \tag{A2.1}$$

where ϕ is unique and ζ is unique up to a Killing vector field of γ. Since the Lie derivatives of all the equations for $\mathcal{E}$ simply give the linearized equations for variations of the form $\mathcal{L}_\zeta f$ for $f\in\mathcal{E}$, and since $\mathcal{L}_\zeta W=0$ (we are on the spherical background) we compute the variations of (1.59–1.62, 1.32) assuming that

$$\delta\gamma_{AB}=\phi_{AB}, \quad \nabla_B\phi^B_A=0. \tag{A2.2}$$

Since the central potential and pressure are fixed, $\delta\tilde{p}$ is given by (4.10) with $k=0$. When S_c is a Euclidean sphere (so $\partial_A\Omega=0$), equation (1.61) together with lemma A1.3 implies that $\Omega_{AB}=\frac{1}{2}\Omega\gamma_{AB}$. (Recall also that $\Omega=2WD(\log r)$ and $\overline{R}=2r^{-2}$.) The linearized Einstein equations (1.59–1.62) are now found to be

$$(MW^{-1}+\Omega)w+WDw+\tfrac{1}{2}W^2D\phi=0 \tag{A2.3}$$

$$\Omega W^{-2}\partial_A w+\partial_A D\phi=0 \tag{A2.4}$$

$$(\overline{\Delta}+\tfrac{1}{2}\overline{R})\phi+\tfrac{1}{2}W\Omega D\phi=-2W^{-1}(\overline{R}+2\tilde{p})w \tag{A2.5}$$

$$\overline{\Delta}(W^{-1}w)-W\Omega D(W^{-1}w)-4\tilde{p}W^{-1}w-\tfrac{1}{2}W^2DD\phi-\tfrac{1}{2}MD\phi=0. \tag{A2.6}$$

Using (A2.3) and its D derivative in (A2.6) one can calculate that

$$\bar{\Delta} w+W^2DDw+(3M-2W^2D(\log r))Dw+(2DM-2M^2W^{-2}+8MD(\log r)-4\tilde{p})w=0. \tag{A2.7}$$

The three-dimensional Laplacian Δ has in the spherically symmetric case the form

$$\Delta f=W^2DDf+MDf+\bar{\Delta} f \tag{A2.8}$$

for any function f on Σ. After some calculations (A2.7) can be seen to be equivalent to

$$\Delta\tilde{w}=F\tilde{w} \tag{A2.9}$$

where $\tilde{w}=m(U)r^{-1}w$, $m(U)=2\int_{U_c}^{U}Mr^4dU$ ($m(U)$ agrees with the gravitational mass m in vacuo) and

$$F(U)=-DM+2M^2W^{-2}-6MD(\log r)+3\tilde{p}+W^2. \tag{A2.10}$$

Now, $(\mathcal{L}_\xi\gamma_{AB})(U)\in M^p_{s-1,\delta+1}$ since the discontinuities in derivatives of γ_{AB} occur only in the normal derivatives $D\gamma_{AB}$. Thus $\phi(U)\in M^p_{s-1,\delta+1}$ and (A2.3) then implies that $w(U)\in M^p_{s-1,\delta+1}$ so $\tilde{w}\in M^p_{s-1,2}\subset C^1$. If $F(U)$ were everywhere positive it would follow immediately from (A2.9) that $\tilde{w}$ vanishes. However this is not quite true due to the only negative term $-6MD(\log r)$, which makes F negative near the center.

To make a closer analysis, let

$$\tilde{w}=\sum_{\ell\geq 0}K_\ell(U)Y_\ell(\theta,\varphi) \qquad \text{(A2.11)}$$

where Y_ℓ is any normalized linear combination of spherical harmonics of order ℓ so they satisfy $\Delta^* Y_\ell+\ell(\ell+1)Y_\ell=0$ and $\int_{S^2}Y_\ell^2 d\Omega=1$ where Δ^* is the Laplacian restricted to the unit sphere. Then $\overline{\Delta}=r^{-2}\Delta^*$ and (A2.9) reduces to

$$W^2r^2DDK_\ell+Mr^2DK_\ell=[r^2F+\ell(\ell+1)]K_\ell, \quad \ell=0,1,2,\cdots \qquad \text{(A2.12)}$$

We first investigate this equation near the center. Since w is C^1, $\tilde{w}$ vanishes at the center and in fact must vanish at least like $O(z^2)$ where $z=(1/6)M_c(U-U_c)^{1/2}$. Since all our functions are regular at the origin we can expand in terms of at least a few powers of z. Comparing the first coefficients of the powers of z in (A2.12) shows that $K_\ell(z)=a_{0\ell}z^\ell+O(z^{\ell+1})$ near the center so that $K_0=K_1=0$.

The remaining multipoles can be shown to vanish by showing that

$$F_\ell=F+\ell(\ell+1)r^{-2}>0 \quad \text{for all } \ell\geq 2 \qquad \text{(A2.13)}$$

everywhere on Σ. For then (A2.12) implies $\Delta K_\ell^2=2K_\ell\Delta K_\ell+2W^2(DK_\ell)^2$ $=2F_\ell K_\ell^2+2W^2(DK_\ell)^2\geq 0$ so that K_ℓ^2 can not have a maximum on Σ. Since $\lim_{r\to\infty}K_\ell=0$ this implies that $K_\ell=0$ on all of Σ so $w=0$. That ϕ_{AB} then vanishes can be seen as follows. Equations (A2.3) and (A2.5) with $w=0$ imply that $\overline{\Delta}\phi+r^{-2}\phi=0$ or, restricted to any sphere S_c, $\Delta^*\phi+\phi=0$. Since the first nonzero eigenvalue of Δ^* on a 2-sphere is -2 there are no regular solutions of this equation so it follows that $\phi=0$. Since $\nabla^A\phi_{AB}=0$, lemma A1.3 implies that $\phi_{AB}=0$.

It thus remains only to show (A2.13). Let

$$u=mr^{-1},\quad v=\tilde{p}r^2,\quad x=\tfrac{1}{3}Mr^2u^{-1}. \tag{A2.14}$$

Then $0\le p\le \rho$ implies $0\le\frac{2}{3}v\le xu$ and (A2.13) is implied by

$$Fr^2\ge 18x^2+15ux+u^2-18x\sqrt{1+u^2+v}\ge -6. \tag{A2.15}$$

This in turn is implied by

$$\begin{aligned}&(324x^4-108x^2+36)+x^2(486ux-324v)\\&+(30u^3x-63x^2u^2+54ux^3)+u^4+180ux+12u^2\ge 0.\end{aligned} \tag{A2.16}$$

The first term is just $(18x^2-6)^2+108x^2\ge 0$ while the second is $486u^2(ux-v2/3)\ge 0$ and the third is greater than $ux(\sqrt{54}x-\sqrt{30}u)^2+18u^2x^2\ge 0$, which proves the relation (A2.13).

APPENDIX 3: LINEARIZATION OF THE (2+1)-DIMENSIONAL STATIONARY FIELD EQUATIONS ON THE SPHERICAL BACKGROUND

The equations which we present here were first given by Künzle and Savage(1980a). They are mainly of interest because of their similarity to the linearized 3-dimensional equations (on the spherical background) involving δh^1 and the linearized static equations in appendix 2.

From §1.4 it is seen that stationary, rigid, perfect fluid spacetimes with a fixed equation of state, fixed central potential U_c and fixed central pressure p_c can be characterized by the set $\mathcal{F}=\{(\bar{\gamma},\Omega,W,h^0,\bar{h},p,\bar{\theta},T)\}$ where all these tensors are functions of $U\in(U_c,0)$ and where we drop the bars from now on when there is no danger of confusion. We write $\delta W=w$, $\delta h^0=k$, $\delta h_A=\kappa_A$, $\delta\theta^A=\vartheta^A$, $\delta T=\tau$ and as in appendix 2 we use the Berger–Ebin decomposition to write

$$\delta\gamma_{AB}=\phi_{AB}+\mathcal{L}_\zeta\gamma_{AB}, \quad \nabla_B\phi_A^B=0 \qquad \text{(A3.1)}$$

where ϕ is unique and ζ is unique up to a Killing vector field of γ. Again we can eliminate the coordinate freedom corresponding to Lie derivatives of the field equations with respect to the vector field ζ and compute the variations of (1.48–1.58) assuming that

$$\delta\gamma_{AB}=\phi_{AB}, \quad \nabla_B\phi^B_A=0. \qquad \text{(A3.2)}$$

A Riemannian 2-sphere S has no closed two-dimensional subgroup of its isometry group I, and is isometric to the Euclidean 2-sphere if $\dim I=3$ (c.f. Kobayashi 1972 p.47). Therefore, in the case $\dim I<3$, we have that inside matter $\zeta^A=c\theta^A$ for some $c\in\mathbb{R}$. But all the quantities are invariant under θ^A so all variations will then vanish under a coordinate transformation. Outside matter θ^A and τ are no longer defined, but we have assumed axisymmetry, so all variations will vanish under a coordinate transformation. If S is isometric to the Euclidean 2-sphere (for all U), all quantities become functions of U only, $\theta^A=h^A=0$ and all variations will vanish under a coordinate transformation.

On a spherical background we have $\theta^A=h^A=h^0=0$ and as in appendix 2, $\Omega_{AB}=\frac{1}{2}\Omega\gamma_{AB}$. Linearization of (1.57) gives $D\tau+\tau=0$ $=\partial_A\tau$ whence $\tau=\tau_s e^{U_s-U}$ which vanishes since we are restricting to models with the same surface temperature T_b. Note also that the integration constant c appearing in $\delta p=(\rho+p)(T^{-1}\tau-c)$ must then be zero (recall $\rho\in M^p_{s-3,n}$ but $\delta p\in M^p_{s-2,n}$) since we keep the central pressure fixed under the variation. Using these facts we now find for the linearized Einstein equations (1.48–1.53)

$$(MW^{-1}+\Omega)w+WDw+\tfrac{1}{2}W^2D\phi=0, \qquad \text{(A3.3)}$$

$$\Omega W^{-2}\partial_A w+\partial_A D\phi=0, \qquad \text{(A3.4)}$$

$$(\overline{\Delta}+\tfrac{1}{2}\overline{R})\phi+\tfrac{1}{2}W\Omega D\phi=-2W^{-1}(\overline{R}+2\tilde{p})w, \qquad \text{(A3.5)}$$

$$\overline{\Delta}(W^{-1}w)-W\Omega D(W^{-1}w)-4\tilde{p}W^{-1}w-\tfrac{1}{2}W^2DD\phi-\tfrac{1}{2}MD\phi=0, \qquad \text{(A3.6)}$$

$$D\kappa_A+4\kappa_A-W^{-2}\partial_A k-2W^{-1}(\rho+p)e^{-6U}T_c e^{U_c}\varepsilon_{AB}\vartheta^B=0, \qquad \text{(A3.7)}$$

$$\partial_{[A}\kappa_{B]}=0, \qquad \text{(A3.8)}$$

where $\phi=\gamma^{AB}\phi_{AB}$. Linearization of the remaining equilibrium conditions (1.55) yields

$$\nabla_{(A}\vartheta_{B)}=0 \quad \text{and} \quad D\vartheta^A=0, \qquad \text{(A3.9)}$$

when $\rho\neq0$, and similarly, linearization of $\nabla_r h^r=0$ gives

$$Dk+(2\Omega W^{-1}-MW^{-2})k+\nabla_A\kappa^A=0. \qquad \text{(A3.10)}$$

Equations (A3.3–A3.6) are just those obtained in appendix 2 for the static case while (A3.7–A3.10) are solved in chapter 5.